ELEMENTARY EQUIVALENCE IN POSITIVE LOGIC VIA PRIME PRODUCTS

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A map $f\colon M \to N$ between two structures M and N is said to be a *homomorphism* when for every atomic formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in M$,

$$M \vDash \varphi(a_1, \dots, a_n) \text{ implies } N \vDash \varphi(f(a_1), \dots, f(a_n)).$$

Positive model theory is the part of model theory that deals with the formulas that are preserved by homomorphisms (see, e.g., [9, 10, 11, 14, 15, 16]). It is well known that these coincides with the *positive formulas*, i.e., formulas built from atomic formulas and $\bot$ using only $\exists$, $\wedge$, and $\vee$.

The Keisler-Shelah's Isomorphism Theorem states that two structures are elementarily equivalent if and only if they have isomorphic ultrapowers. This celebrated result was first proved by Keisler under the generalized continuum hypothesis (GCH) [4, Thm. 2.4]. This assumption was later shown to be redundant by Shelah [12, p. 244]. The aim of this talk is to prove a version of Keisler's original theorem in the context of positive model theory.

To this end, we say that two structures are *positively equivalent* when they have the same positive theory. Then we introduce a generalization of the ultraproduct construction that captures positive equivalence. We term this construction a *prime product* because it is obtained by replacing the index set I of an ultraproduct by a poset $\mathbb{X}$ and the ultrafilter over I by a *prime* filter of the bounded distributive lattice of upsets of the poset $\mathbb{X}$. The case of traditional ultraproducts is then recovered by requiring the order of $\mathbb{X}$ to be the identity relation.

Prime products and positive formulas are connected by a positive incarnation of Łoś' Theorem. As a consequence, prime products preserve not only positive formulas, but also the universal closure of the implications between them, known as *basic h-inductive sentences* [11]. This allows us to describe the classes of models of h-inductive theories as those closed under isomorphisms, prime products, and ultraroots.

Our main result states that under GCH two structures have the same *positive theory* if and only if they have isomorphic *prime powers* of ultrapowers. The same result holds without GCH, provided that prime powers are replaced by *prime products* in the statement. Notably, the presence of ultrapowers cannot be removed from these theorems, as there exists positively equivalent structures without isomorphic prime powers. The results of this talk have been collected in the paper [8].

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