# Bi-Intermediate Logics of Trees and Co-trees

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*Bi-intuitionistic logic* bi-IPC is the conservative extension of intuitionistic logic IPC obtained by adding a new binary connective $\leftarrow$ to the language, called the *co-implication* (or exclusion, or subtraction), which behaves dually to $\to$. In this way, bi-IPC achieves a symmetry, which IPC lacks, between the connectives $\wedge, \top, \to$ and $\vee, \bot, \leftarrow$, respectively.

The Kripke semantics of bi-IPC [25] provides a transparent interpretation of co-implication: given a Kripke model $\mathfrak{M}$, a point $x$ in $\mathfrak{M}$, and formulas $\phi, \psi$, then

$$\mathfrak{M}, x \models \phi \leftarrow \psi \iff \exists y \leq x \ (\mathfrak{M}, y \models \phi \text{ and } \mathfrak{M}, y \not\models \psi).$$

Equipped with this new connective, bi-IPC achieves significantly greater expressivity than IPC. For instance, if the points of a Kripke frame are interpreted as states in time, the language of bi-IPC is expressive enough to talk about the past, something that is not possible in IPC. With this example in mind, Wolter extended Gödel's interpretation of IPC into S4 to an interpretation of bi-IPC into tense-S4 [30]. In particular, he proved a version of the Blok-Esakia Theorem [6, 13] stating that the lattice $\Lambda$(bi-IPC) *bi-intermediate logics* (i.e., consistent axiomatic[1] extensions of bi-IPC) is isomorphic to that of consistent normal tense logics containing Grz.t, see also [9, 28].

The greater symmetry of bi-IPC with respect to IPC is reflected in the fact that bi-IPC is algebraized in the sense of [7] by the variety bi-HA of *bi-Heyting algebras* [24], i.e., Heyting algebras whose order duals are also Heyting algebras. As a consequence, the lattice $\Lambda$(bi-IPC) is dually isomorphic to that of nontrivial varieties of bi-Heyting algebras. The latter, in turn, is amenable to the methods of universal algebra and duality theory because the category of bi-Heyting algebras is dually isomorphic to that of *bi-Esakia spaces* [12], see also [3].

The theory of bi-Heyting algebras was developed in a series of papers by Rauszer and others motivated by the connection with bi-intuitionistic logic. However, bi-Heyting algebras arise naturally in other fields of research as well such as topos theory [20, 21, 26]. Furthermore, the lattice of open sets of an Alexandrov space is always a bi-Heyting algebra, and so is the lattice of subgraphs of an arbitrary graph (see, e.g., [29]) and, similarly, every quantum system can be associated with a complete bi-Heyting algebra [11].

The lattice $\Lambda$(IPC) of intermediate logics (i.e., consistent extensions of IPC) has been thoroughly investigated (see, e.g., [8]). On the other hand, the lattice $\Lambda$(bi-IPC) of bi-intermediate logics lacks such an in-depth analysis, but for some recent developments see, e.g., [1, 4, 14, 15, 27]. In this paper we shall contribute to fill this gap by studying a simpler, yet nontrivial, sublattice of $\Lambda$(bi-IPC): the lattice of consistent extensions of the *bi-intuitionistic linear calculus* (or the bi-Gödel-Dummett's logic),

$$\mathsf{bi\text{-}LC} \coloneqq \mathsf{bi\text{-}IPC} + (p \to q) \vee (q \to p).$$

[1] From now on we will use *extension* as a synonym of *axiomatic extension.*

Notably, the properties of $\Lambda(\mathsf{bi\text{-}IPC})$ and its extensions diverge significantly from those of its intermediate counterpart, i.e., the *intuitionistic linear calculus* (or the Gödel-Dummett's logic) $\mathsf{LC} \coloneqq \mathsf{IPC} + (p \to q) \vee (q \to p)$ [10, 16].

The choice of $\mathsf{bi\text{-}LC}$ as a case study was motivated by some of its properties that make it an interesting logic on its own. On the one hand, $\mathsf{bi\text{-}LC}$ is complete in the sense of Kripke semantics with respect to the class of *co-trees* (i.e., order duals of trees). Moreover, we prove that the bi-intuitionistic logic of linearly ordered Kripke frames is a proper extension of $\mathsf{bi\text{-}LC}$. This contrasts with the case of intermediate logics, where $\mathsf{LC}$ is both the logic of the class of linearly ordered Kripke frames and of co-trees. Because of this, the language of $\mathsf{bi\text{-}IPC}$ seems more appropriate to study tree-like structures than that of $\mathsf{IPC}$. Furthermore, because of the symmetric nature of bi-intuitionistic logic, our results on extensions of $\mathsf{bi\text{-}LC}$ can be extended in a straightforward manner to the extensions of the bi-intermediate logic of trees by replacing in what follows every formula $\varphi$ by its dual $\neg\varphi^\partial$, where $\varphi^\partial$ is the formula obtained from $\varphi$ by replacing each occurrence of $\wedge, \top, \to$ by $\vee, \bot, \leftarrow$ respectively, and every algebra or Kripke frame by its order dual.

On the other hand, the logic $\mathsf{bi\text{-}LC}$ admits a form of a classical reductio ad absurdum, as we proceed to explain. A deductive system $\vdash$ is said to have a *classical inconsistency lemma* if, for every nonnegative integer $n$, there exists a finite set of formulas $\Psi_n(p_1, \dots, p_n)$, which satisfies the equivalence

$$\Gamma \cup \Psi_n(\varphi_1, \dots, \varphi_n) \text{ is inconsistent in } \vdash \iff \Gamma \vdash \{\varphi_1, \dots, \varphi_n\}, \tag{1}$$

for all sets of formulas $\Gamma \cup \{\alpha_1, \dots, \alpha_n\}$ [23] (see also [19, 18]). As expected, the only intermediate logic having a classical inconsistency lemma is $\mathsf{CPC}$ (with $\Phi_n \coloneqq \{\neg(p_1 \wedge \cdots \wedge p_n)\}$). This is far from the case of bi-intermediate logics. For example, we prove that every member of $\Lambda(\mathsf{bi\text{-}LC})$ has a classical inconsistency lemma witnessed by

$$\Phi_n \coloneqq \{\sim\neg\sim(p_1 \wedge \cdots \wedge p_n)\},$$

where $\neg p$ and $\sim p$ are shorthand for $p \to \bot$ and $\top \leftarrow p$ (see, e.g., [22, Chpt. 4]). Accordingly, logics in $\Lambda(\mathsf{bi\text{-}LC})$ exhibit a certain balance between the classical and intuitionistic behavior of negation connectives.

The logic $\mathsf{bi\text{-}LC}$ is algebraized by the variety $\mathsf{bi\text{-}GA}$ of *bi-Gödel algebras*, i.e., the class of bi-Heyting algebras which satisfy Gödel's pre-linearity axiom $(p \to q) \vee (q \to p)$. This is a semi-simple variety of bi-Heyting algebras, hence it follows from [29] that it has a discriminator term, and therefore has EDPC. Moreover, as this variety is axiomatized (relative to $\mathsf{bi\text{-}HA}$) by a $\leftarrow$-free formula and has a locally finite Heyting algebra reduct [8], it follows from [22, Chpt. 3] that $\mathsf{bi\text{-}GA}$ enjoys the finite model property.

As for the geometric models of $\mathsf{bi\text{-}LC}$, these take the form of *bi-Esakia co-forests*, i.e., bi-Esakia spaces whose underlying posets are disjoint unions of co-trees. In particular, the dual spaces of the simple bi-Gödel algebras are termed *bi-Esakia co-trees*, and as finite bi-Esakia spaces are equipped with the discrete topology, all finite co-trees can viewed as a bi-Esakia co-trees.

The main contributions of our work can be summarized as follows. We develop a theory of Jankov, subframe and canonical formulas of bi-Gödel algebras. We employ Jankov formulas to obtain a characterization of splittings in $\Lambda(\mathsf{bi\text{-}LC})$ and canonical formulas to uniformly axiomatize all the extensions of $\mathsf{bi\text{-}LC}$, cf. [2].

**Theorem 1.** *If $L \in \Lambda(\mathsf{bi\text{-}LC})$, then:*

1. *$L$ is a splitting logic iff $L$ is the logic of a finite co-tree;*

2. *$L$ is axiomatizable by canonical formulas. Moreover, if $L$ is finitely axiomatized, then $L$ is axiomatizable by finitely many canonical formulas.*

We also use Jankov formulas to show that $\Lambda(\mathsf{bi\text{-}LC})$ has the cardinality of the continuum. This is achieved by means of the construction of an infinite antichain (with respect to the order of being a bi-Esakia morphic image) of finite co-trees[2], and contrasts with the case of $\Lambda(\mathsf{LC})$ which is well known to be a chain of order type $(\omega+1)^{\partial}$ [8].

Lastly, subframe formulas can be used to describe the fine structure of co-trees, since a bi-Esakia co-tree $\mathcal{X}$ refutes the subframe formula of (the algebraic dual of) a finite co-tree $\mathfrak{F}$ iff $\mathcal{X}$ admits $\mathfrak{F}$ as a subposet. For the present purpose, the interest of subframe formulas is that they help us characterize the locally tabular extensions of $\mathsf{bi\text{-}LC}$. This is done in three steps, all relying on the structure of the *finite combs*, a particular class of co-trees depicted in Figure 1.

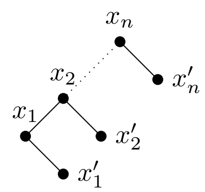


Figure 1: The $n$-comb $\mathfrak{C}_n$

Firstly, we prove that for all positive integers $n$, a bi-Esakia co-tree $\mathcal{X}$ admits $\mathfrak{C}_n$ as a subposet iff $\mathfrak{C}_n$ is a bi-Esakia morphic image of $\mathcal{X}$. Secondly, we find a natural bound for the size of $m$-generated simple bi-Gödel algebras whose bi-Esakia duals do not admit the $n$-comb $\mathfrak{C}_n$ as a subposet. Finally, by showing that the variety generated by the (algebraic duals of the) finite combs is not locally finite, we derive the following criterion and an immediate corollary:

**Theorem 2.** *If $L \in \Lambda(\mathsf{bi\text{-}LC})$, then $L$ locally tabular iff $\mathfrak{C}_n$ is not a model of $L$, for some $n \in \omega$.*

**Corollary 3.** *A variety $\mathsf{V}$ of bi-Gödel algebras is locally finite iff $\mathsf{V}$ omits the algebraic dual of a finite comb. Consequently, the variety generated by the duals of the finite combs is the only pre-locally finite variety of bi-Gödel algebras.*

It follows from above that $\mathsf{bi\text{-}LC}$ is not locally tabular, highlighting yet another contrast with $\mathsf{LC}$, which is well known to be locally tabular [17]. These results are collected in [5].

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[2]The authors would like to thank I. Hodkinson for his help in constructing this antichain.

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