Bounded distributive lattices with a weak strict implication

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A monotonic neighbourhood frame is a pair $\langle X, N\rangle$ where X is a non-empty set, and N is a relation between X and $\mathcal{P}(X)$ such that for each $x \in X$, if $U \subseteq V$ and $U \in N(x)$, then $V \in N(x)$. By means of the relation N we define the operators:

$$m_N(U) = \{x \in X : U \in N(x)\},$$

and

$$\Box_N(U) = \{x \in X : \forall Y \in N(x)\,(Y \subseteq U)\},$$

The algebras of the form $\langle \mathcal{P}(X), m_N\rangle$ generate the variety MonBA of monotone modal algebras [2] [4], and the algebras of the form $\langle \mathcal{P}(X), \Box_N\rangle$ generate the variety MA of modal algebras. By means of these operators we can define two *like* strict implications on $\mathcal{P}(X)$ as:

$$U \mapsto_N V := m_N(U^c \cup V), \tag{1}$$

and

$$U \Rightarrow_N V = \Box_N(U^c \cup V), \tag{2}$$

for all $U, V \in \mathcal{P}(X)$. Since $m_N(U) = X \mapsto_N U$ and $\Box_N(U) = X \mapsto_N U$, for every $U \in \mathcal{P}(X)$, the modal operator m_N is interdefinable with $\mapsto_N$, and the modal operator $\Box_N$ is interdefinable with $\Rightarrow_N$.

The algebra $\langle \mathcal{P}(X), \mapsto_N\rangle$ is a *monotone Lewis algebra* [3, Definition 7.2]. As it is proved in [3], the variety MonBA is equivalent to the variety of monotone Lewis algebras MLA. An important fact about the algebras $\langle \mathcal{P}(X), \Rightarrow_N\rangle$ is that although $\Box_N$ is a normal modal operator, the implication $\Rightarrow_N$ is not a subintuitionistic implication, i.e., $\langle \mathcal{P}(X), \Rightarrow_N\rangle$ is not a weak Heyting algebra.

So, we can formulate two problems:

Problem 1 Study the $\{\vee, \wedge, \mapsto, \bot, \top\}$-subreducts of the variety MLA.

Problem 2 Study the $\{\vee, \wedge, \Rightarrow, \bot, \top\}$-subreducts of the variety of algebras generated by the algebras of the form $\langle \mathcal{P}(X), \Rightarrow_N\rangle$.

The problem **Problem 1** is studied in [3]. The $\{\vee, \wedge, \mapsto, \bot, \top\}$-subreducts of monotone Lewis algebras is the variety MLD of distributive lattices with monotone implication [3, Definition 3.6]. The variety MLD is the algebraic semantic of a weak subintuitionist logic $\mathcal{P}_{\mapsto}$ defined in [3],

Addressing the problem Prob 2 is the main goal of this talk. A We present the variety generated by the $\{\vee, \wedge, \Rightarrow, \bot, \top\}$-subreducts of the variety generated by the algebras $\langle \mathcal{P}(X), \Rightarrow_N\rangle$. We study its representation by means of ordered neighbourhood frames.

References

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