TACL 2024

Book of abstracts

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Preface

This volume contains the papers presented at the conference **TACL 2024: Topology, Algebra, and Categories in Logic** held in Barcelona on July 1–5, 2024. The volume includes the abstracts of 129 accepted contributed talks and of 12 invited talks.

TACL 2024 is the 11^{th} conference in the series Topology, Algebra, and Categories in Logic (TACL, formerly TANCL). Earlier instalments were held in Tbilisi (2003), Barcelona (2005), Oxford (2007), Amsterdam (2009), Marseille (2011), Nashville (2013), Ischia (2015), Prague (2017), Nice (2019), and Coimbra (2022). The conference series focuses on three interconnecting mathematical themes central to the semantic study of logic and its applications: topological, algebraic, and categorical methods.

We thank all Programme Committee members for their precious work in reading the submitted abstracts and giving useful suggestions to the authors.

Barcelona, June 2024

The organizers

Contents

Invited talks

Contributed talks

Invited talks

An introduction to Π^1_2 Proof Theory

Juan Aguilera

Vienna University of Technology, Austria

We will give a gentle introduction and survey to the world of Π^1_2 Proof Theory, in particular focusing on the notion of a dilator and on some of its uses.

Arrow algebras

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Arrow algebras are algebraic structures which can be used to construct toposes. These "arrow toposes" include both localic toposes (toposes obtained from a locale, i.e. a complete Heyting algebra) and realizability toposes obtained from partial combinatory algebras. In that way arrow algebras are similar to Alexandre Miquel's implicative algebras, which were the main source of inspiration for the concept. However, the notion of an arrow algebra is weaker and this weakening is motivated by the desire to include more examples and to have a better interaction with nuclei. In this talk I will explain this (which is joint work with Marcus Briet) and also report on work by Umberto Tarantino who has looked into the question of what would be a good notion of morphism of arrow algebras.

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On quantale enriched monoids

Célia Borlido

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Highlighting the analogy between the triangle inequality for metric spaces and the categorical composition law, in 1973, Lawvere argued that "fundamental structures [in mathematics] are themselves categories" and thus, the latter should not be treated as a kind of "third level of abstraction". Accordingly, he developed a framework in which a (generalized) metric space may be seen as a category enriched in the real half-line extended with the infinity value. More generally, for a quantale $\mathcal{V}$, a $\mathcal{V}$-category may be defined as a set equipped with a binary $\mathcal{V}$-valued relation satisfying reflexivity and transitivity axioms. For different choices of the quantale $\mathcal{V}$, we then obtain different mathematical structures, including preordered sets, the already mentioned Lawvere's metric spaces, and probabilistic metric spaces.

In this talk we will consider monoids equipped with a compatible structure of $\mathcal{V}$-category, to which we call $\mathcal{V}$-monoids, and discuss some of their properties. In particular, we will investigate the possible quantale enrichments on semidirect products of $\mathcal{V}$-monoids as well as their connections to split extensions.

Extending the Blok-Esakia Theorem to the monadic setting

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It is a classic result of McKinsey and Tarski [6] that the Gödel translation embeds the intuitionistic propositional calculus IPC into the propositional modal logic $\mathsf{S4}$. The normal extensions of $\mathsf{S4}$ into which the Gödel translation embeds a given superintuitionistic logic L are called the *modal companions* of L. Esakia's Theorem [3] states that the largest modal companion of IPC is the Grzegorczyk logic $\mathsf{Grz} \coloneqq \mathsf{S4} + \mathsf{grz}$, where

$$\mathsf{grz} = \Box(\Box(p \to \Box p) \to p) \to p.$$

Every superintuitionistic logic L has a least and a greatest modal companion, denoted $\tau\mathsf{L}$ and $\sigma\mathsf{L}$, and hence the modal companions of L form an interval $[\tau\mathsf{L}, \sigma\mathsf{L}]$ in the lattice of normal extensions of $\mathsf{S4}$. The celebrated Blok-Esakia Theorem, established independently by Blok [2] and Esakia [3], states that mapping L to $\sigma\mathsf{L}$ yields an isomorphism between the lattice of superintuitionistic logics and the lattice of normal extensions of Grz.

The predicate extension of the Gödel translation embeds the intuitionistic predicate calculus IQC into the predicate $\mathsf{S4}$ logic $\mathsf{QS4}$. However, the behavior of modal companions of predicate superintuitionistic logics is much less understood. It is convenient to first investigate the restriction of the predicate Gödel translation to the *monadic fragments* (also known as the *one-variable fragments*) MIPC of IQC and $\mathsf{MS4}$ of $\mathsf{QS4}$, which can be thought of as bimodal logics and can be studied using the standard semantic tools of modal logic. Fischer Servi [5] proved that the Gödel translation embeds MIPC into $\mathsf{MS4}$ and Esakia [4] showed that the monadic Grzegorczyk logic $\mathsf{MGrz} \coloneqq \mathsf{MS4} + \mathsf{grz}$ is a modal companion of MIPC. It is then natural to wonder whether an analogue of the Blok-Esakia Theorem holds in the monadic setting.

This talk will address the challenges involved in the study of modal companions of extensions of MIPC. We will see that the Blok-Esakia Theorem fails in the monadic setting: the map from the lattice of extensions of MIPC to the lattice of extensions of MGrz that naturally generalizes σ is not a lattice isomorphism. We will then discuss the obstacles to the generalization of Esakia's Theorem to MIPC. Some possible ways to recover positive results in the monadic setting will also be mentioned. This talk is based on joint work with G. Bezhanishvili and most of the discussed results can be found in [1].

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Relational semantics and ordered algebras for monotone propositional logics

Ramón Jansana

I will present part of the general theory of relational semantics for propositional logics developed together with Tommaso Moraschini in the recent past. More specifically, I will expound a relational semantics for monotone logics.

A logic (as a consequence relation) is monotone if every connective is in each coordinate either increasing or decreasing with respect to the pre-order induced in the algebra of formulas by the consequence relation (i.e., the pre-order that declares a formula below another if the second follows from the first.)

The semantics we develop is based on a duality between a class of ordered algebras associated with a monotone logic and a class of general frames for it. The idea we use to turn an ordered algebra into a frame and conversely is inspired by the notion of the relational dual of a function, coming from B. Jónsson and A. Tarski's work on Boolean algebras with operators, as well as from M. Dunn's gaggle theory; it also is inspired by M. Gehrke's work on RS-frames.

The frames we use consist of a polarity–a set of positive states (worlds), a set of negative states (co-worlds), and a relation between them–and for each connective of the language of the logic a suitable relation between worlds and co-worlds in accordance to the logical behavior of the connective.

We will discuss some examples and see that in many well-known cases our relational semantics specializes to the traditional one.

Duality theory for Boolean right restriction semigroups

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Institute of Mathematics, Physics and Mechanics in Ljubljana

We generalize the duality between Boolean right restriction monoids and ample source-etale topological categories by Cockett and Garner to the non-unital and locally compact setting. Our approach stems from the widely known construction of the tight groupoid of an inverse semigroup as the groupoid of germs. Elements of a supported Boolean right restriction semigroup are represented by right compact slices of their attached right ample categories. In these categories, the domain map is a local homeomorphism, but the range map is not open in general, and thus does not give rise to a unary range operation in the associated right restriction semigroup. In the special case where the range map of the right ample category is open, the associated right restriction semigroup has the additional structure of a left Ehresmann semigroup. Specializing further to the case where the range map is a local homeomorphism, the category has the additional property that every compact right slice is a finite join of compact two-sided slices. On the algebraic side, this brings Boolean right restriction Ehresmann semigroups with the extra property that every element is a finite join of deterministic elements. These semigroups are a natural generalization of Garner's groupoidal right restriction monoids.

Being, Becoming, and the dimension of combinatorial spaces

Matías Menni

CONICET & Universidad Nacional de La Plata

Lawvere's 1990 Thoughts of the Future of Category Theory [2] outlines a positive mathematical programe; an "attempt, by an admirer of rational mechanics, to include objective logic among the tools for arriving at a more accurate conception of space". We concentrate here on two of the philosophical guides there, and a conjecture relating them.

Partially motivated by the opposition between 'gros' and 'petit' in Algebraic Geometry, the first philosophical guide is that there is a distinction between a *general category of Being* and *particular categories of Becoming*. The terminology reflects the "hope that sober application, of category theory to the ancient philosophical categories, will not only clarify both but also renew respect for serious thought, through solid examples approaching adequacy to their concept." Alternatively, we may phrase the distinction as one between toposes 'of spaces' [3] on the one hand and 'generalized locales' on the other. Additionally, the philosophical guide includes a way to relate the two classes of toposes: each space X (i.e. an object in a category of spaces) should determine a generalized locale $\mathrm{P}(X)$ of 'pseudo-classical sheaves' on X.

The second philosophical guide is that each topos of spaces determines a poset of dimensions which may be identified with the poset of *levels* (i.e. essential subtoposes) of the given topos.

The conjecture relating the two guides is that the dimension of a space X depends only on the generalized locale $\mathrm{P}(X)$. "In other words, if we have an equivalence of categories $\mathrm{P}(X) \cong \mathrm{P}(Y)$, then X, Y should belong to the same class of UIO levels within the category of Being in which they are objects. Suitable hypotheses to make this conjecture true should begin to clarify the relationships between the two suggested philosophical guides."

We show that the conjecture holds for a solid class of examples of presheaf toposes including that of simplicial sets. Moreover, we prove that the way in which $\mathrm{P}(X)$ determines the dimension of X is almost identical to that in which the algebra of open subpolyhedra of a polyhedron determines its dimension [1].

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Refining Intentional Modals via Topology

Aybüke Özgün

University of Amsterdam

This talks provides an overview of two applications of topological semantics in philosophical logic: (1) topological semantics for epistemic logic based on possible worlds and (2) topological semantics for modeling topic-sensitive (binary) intentional modals based on an algebra of topics.

The first part of this talk will be concerned with modal logics for evidence, knowledge, and belief. The traditional treatment of epistemic logics based on relational semantics is not rich enough to talk about the evidential nature of acquired knowledge and belief. I will argue that topological spaces emerge naturally as information structures if one not only seeks an easy way of modeling knowledge and belief, but also aims at representing evidence and its relationship to these notions. Based on the semantics proposed in [1], I will show that the topological approach enables fine-grained and refined representations of the aforementioned epistemic notions, highlighting several variations and extensions in the literature. (This part of the talk is based on joint work with Alexandru Baltag, Nick Bezhanishvili, and Sonja Smets).

In the second part of the talk, I will focus on logics of imagination, which formalize the notion of imagination via a binary, topic-sensitive modal operator. In the topic-sensitive theory of the logic of imagination, the topic of the imaginative output must be contained within the topic of the imaginative input. That is, imaginative episodes can never expand what they are about. This constraint is implausible from a psychological point of view, and it wrongly predicts the falsehood of true reports of imagination. I will present a number of direct approaches to relaxing this controversial topic-inclusion constraint. The core idea involves adding an expansion operator to the algebra of topics. The logic that results depends on the formal constraints placed on topic expansion, the choice of which are subject to philosophical dispute. The first semantics I will present is a topological one using a closure operator. I will also explore a few weaker topic expansion operators and their associated logics. Time permitting, I will elaborate on further generalizations of the topic-sensitive semantics of imagination and the applications of proposed topic expansion operators to knowledge, belief, and conditionals. (This part of the talk is based on joint work with Aaron J. Cotnoir)

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Priestley-type dualities beyond the case of finite dualizing objects

Adam Přenosil

University of Barcelona

Categorical dualities between classes of algebras and classes of topological spaces equipped with relational structure underlie a great number of remarkable results in the field of algebraic logic and beyond. A systematic account covering many such dualities exists in the form of the theory of natural dualities, developed over the last decades among others by Davey, Priestley, and Werner. This theory has been worked out in much detail for the case of quasivarieties generated by a finite algebra (which acts as a dualizing object inducing a so-called concrete duality). However, only a fragment of this theory has been developed beyond the case of finite dualizing objects. Taking inspiration from dualities for (weakly) locally finite MV-algebras due to Cignoli, Dubuc, Marra, and Mundici, we establish a Priestley-type duality induced by a possibly infinite dualizing algebra with a near unanimity term (such as the standard MV-chain or the standard positive MV-chain) and showcase some of its applications.

The talk is based on joint work with Marco Abbadini.

Game comonads and resource-sensitive model theory

Luca Reggio

University College London, UK

Since the pioneering work of Lawvere in the 1960s, category theory has been used to provide a syntax-independent view of the fundamental structures of logic, including e.g. first-order logic and extensions to infinitary and higher-order languages.

In this talk, motivated by the needs of finite model theory and descriptive complexity, I will focus on logic fragments that involve a finite amount of *logical resources*, such as finite-variable logics or logics with bounded quantifier rank, and the corresponding combinatorial parameters of (relational) structures. A key insight due to Abramsky, Dawar and their collaborators is that, in many cases of interest, these resource-sensitive logics can be described by comonads on the category of structures, and the associated combinatorial parameters by the coalgebras for the comonads. This is at the origin of the framework of *game comonads* [1, 5].

I will survey the main ideas underlying game comonads and the axiomatic approach of *arboreal categories* [2, 3], and some of their applications. The latter are due to several teams of authors and include:

(i) A categorical view of homomorphism counting results in finite model theory [6, 7].

(ii) The axiomatic study of homomorphism preservation theorems in logic [4].

(iii) The interplay between Gabriel–Ulmer duality and the expressive power of arboreal categories [9].

(iv) A homotopical view of modal logic and the Łoś–Tarski preservation theorem [8].

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Girard quantales, their linear orders, and completely distributive lattices*

Luigi Santocanale

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In the first part of this talk I'll define linear orders valued in a Girard quantale, a variant of the usual notion of metric space whose distance is valued in a quantale. I'll give examples of these structures mostly arising from combinatorics and geometry. In particular, if the quantale $Q_\vee([0,1])$ of sup-preserving endofunctions of the unit interval is considered, linear orders on a set of finite cardinality n valued in it can be identified with images of continuous monotone paths in the n-dimensional cube $[0,1]^n$, linking the origin to the unit vector [3].

Considering that the quantale of sup-preserving endofunctions of a complete lattice is Girard if and only if the lattice is completely distributive (and that complete chains are completely distributive), the linear orders described above have motivated further research on completely distributive lattices, see e.g. [2]. In a second part of this talk I'll focus on constructing completely distributive lattices from given ones, using complete congruences. Indeed, it is an elementary observation that the quotient of a completely distributive lattice by a complete congruence is again completely distributive. In a recent work [1] we have given a geometrical characterisation (as sublocales) of these complete congruences. The characterisation relies on Hoffmann-Lawson duality between completely distributive lattices and their spectra. I'll illustrate the characterisation with the posets $(0,1]$ and $[0,1)^{op} \times (0,1]$, that are the spectra of $[0,1]$ and $Q_\vee([0,1])$, respectively. In particular, I'll argue that complete congruences give rise to a frame, and that such frame is not, in general, Boolean, nor completely distributive.

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*Joint work with Maria João Gouveia, Cédric de Lacroix, Cameron Calk

Projectivity in quasivarieties of logic

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This talk is about some *bridge theorems*, i.e. connections between syntactic features of a logic and the properties of its algebraic semantics, and the role that projective algebras have in them. In particular, we will be concerned with algebraizable logics in the sense of Blok-Pigozzi [3], whose equivalent algebraic semantics are quasivarieties of algebras.

Given a quasivariety Q, an algebra $\mathbf{P}$ is *projective* in Q if and only if it is a retract of a free algebra in Q. Thus, projective algebras extend the class of free algebras, and they can "play the role" of free algebras in some instances as we will discuss. It is important to note that projectivity is a categorical notion, and as such is preserved by categorical equivalence, while free algebras in general are not. This allows the study of projective algebras via categorical equivalences or dualities with respect to objects that are easier to understand.

In the first part of this talk we will present some examples of characterizations of projective algebras in relevant (quasi)varieties related to logic, involving techniques that vary from purely algebraic to duality theoretic. We will present some examples from the literature, and some recent results from [1]. The examples mostly lie in the framework of residuated lattices, which provide the equivalent algebraic semantics of substructural logics; the latter comprise classical logic and many of the most interesting nonclassical logics, e.g. intuitionistic logic, relevance logics, linear logic, many-valued logics.

In the second part of the talk, we discuss the role of projective algebras in the algebraic study of unification problems, in the setting developed by Ghilardi [4], together with applications to the study of the structural completeness of a quasivariety and its weakenings [2].

Finally we present a new approach, based on recent joint work with Tommaso Flaminio, to *equational anti-unification problems*, whose synctactic version was first introduced in the 1970s to study inductive proofs. The key is once again the use of projective algebras. We show that both equational anti-unification problems and their type (i.e., the cardinality of the set of "best" - or least general - solutions) can be studied algebraically; we discuss some relevant examples, e.g., in Boolean algebras, Kleene algebras, Gödel algebras, MV-algebras (the equivalent algebraic semantics of, respectively, classical logic, 3-valued Kleene logic, Gödel-Dummett logic, infinite-valued Łukasiewicz logic).

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Contributed talks

Varieties of MV-monoids and positive MV-algebras

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We investigate MV-monoids and their subvarieties. An *MV-monoid* is an algebra $\langle A, \vee, \wedge, \oplus, \odot, 0, 1\rangle$ where:

- $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice;
- $\langle A, \oplus, 0\rangle$ and $\langle A, \odot, 1\rangle$ are commutative monoids;
- $\oplus$ and $\odot$ distribute over $\vee$ and $\wedge$;
- for every $x, y, z \in A$,

$$(x \odot y) \oplus ((x \oplus y) \odot z) = (x \oplus (y \odot z)) \odot (y \oplus z);$$
$$(x \odot y) \oplus z = ((x \oplus y) \odot ((x \odot y) \oplus z)) \vee z;$$
$$(x \oplus y) \odot z = ((x \odot y) \oplus ((x \oplus y) \odot z)) \wedge z.$$

Every MV-algebra in the signature $\{\oplus, \neg, 0\}$ is term equivalent to an algebra that has an MV-monoid as a reduct, by defining, as standard, $1 \coloneqq \neg 0$, $x \odot y \coloneqq \neg(\neg x \oplus \neg y)$, $x \vee y \coloneqq (x \odot \neg y) \oplus y$ and $x \wedge y \coloneqq \neg(\neg x \vee \neg y)$. We show that every subdirectly irreducible MV-monoid $\mathbf{A}$ is totally ordered and satisfies the property: for all $x, y \in A$, $x \oplus y = 1$ or $x \odot y = 0$.

Using this result, we investigate the bottom part of the lattice of subvarieties of MV-monoids, characterizing all the almost minimal varieties of MV-monoids as the varieties generated by:

- a reduct of a finite MV-chain of prime order ($\mathbf{Ł}_p^+$);
- the unique MV-monoid $\mathbf{C}_2^\Delta$ on the 3-element chain $0 < \varepsilon < 1$ satisfying $\varepsilon \oplus \varepsilon = \varepsilon$ and $\varepsilon \odot \varepsilon = 0$;
- the dual of $\mathbf{C}_2^\Delta$.

One of the main tools that we used to develop the theory of MV-monoids is the categorical equivalence Γ between unital commutative ℓ-monoids and MV-monoids [1].

A *unital commutative ℓ-monoid* is an algebra $\langle M, \vee, \wedge, +, 1, 0, -1\rangle$ with the following properties:

- $\langle M, \vee, \wedge, +, 0\rangle$ is a commutative ℓ-monoid;
- $-1 + 1 = 0$;
- $-1 \leq 0 \leq 1$;

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- for all $x \in M$ there is $n \in \mathbb{N}$ such that

$$\underbrace{(-1)+\cdots+(-1)}_{n \text{ times}} \leq x \leq \underbrace{1+\cdots+1}_{n \text{ times}}.$$

Thus the relationship between unital commutative ℓ-monoids and MV-monoids is similar to the one between unital abelian ℓ-groups and MV-algebras and we exploit this fact in several statements of our work. We also present a version of Hölder's theorem for unital commutative ℓ-monoids.

Particular examples of MV-monoids are positive MV-algebras, i.e. the $\{\vee, \wedge, \oplus, \odot, 0, 1\}$-subreducts of MV-algebras or, equivalently, the proper subquasivariety of the variety of MV-monoids (MVM), axiomatized relatively to MVM by

$$(x \oplus z \approx y \oplus z \text{ and } x \odot z \approx y \odot z) \implies x \approx y.$$

Positive MV-algebras form a peculiar quasivariety in the sense that, albeit having a logical motivation (being the quasivariety of subreducts of MV-algebras), it is not the equivalent quasivariety semantics of any logic in the sense of [2].

In this cancellative setting, we characterized the varieties of positive MV-algebras as precisely the varieties generated by finitely many reducts of finite nontrivial MV-chains. We also proved that such reducts coincide with the subdirectly irreducible finite positive MV-algebras. Using these results we prove that: a variety of positive MV-algebras is of the form $\mathcal{V}(\mathcal{K}_I)$, where I is a finite subset of $\mathbb{N} \setminus \{0\}$ containing all the divisors of its elements (*divisor-closed subset*) and $\mathcal{K}_I$ is the set of all reducts of MV-chains $\mathbf{Ł}^+_m$ such that $|Ł^+_m| - 1 \in I$.

In conclusion, we present axiomatizations of all the varieties of positive MV-algebras, using a strategy similar to the one of Di Nola and Lettieri [3]. To do so we define the following set of equations.

Let $I \subseteq \mathbb{N}$ be a divisor-closed set, and let m be the maximum of I (with the convention that $m = 0$ if $I = \emptyset$). We define Σ_I as the set of equations given by:

$$(m+1)x \approx mx \qquad \text{and} \qquad m((k-1)x)^k \approx (kx)^m \tag{1}$$

for all $1 \leq k \leq m$ such that $k \notin I$. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ we define the unary term $\tau_{n,k}(x)$ inductively on n as follows:

$$\tau_{0,k}(x) \coloneqq \begin{cases} 1 & \text{if } k \leq -1, \\ 0 & \text{if } k \geq 0, \end{cases} \qquad \tau_{n+1,k}(x) = \tau_{n,k-1}(x) \odot (x \oplus \tau_{n,k}(x)).$$

For every $n \in \mathbb{N}$, let Φ_n be the following set of equations, for k ranging in $\{0, \ldots, n-1\}$:

$$\tau_{n,k}(x) \oplus \tau_{n,k}(x) \approx \tau_{n,k}(x) \text{ and } \tau_{n,k}(x) \odot \tau_{n,k}(x) \approx \tau_{n,k}(x). \tag{2}$$

Theorem. Let I be a divisor-closed finite set; then $\mathcal{V}(\mathcal{K}_I)$ is axiomatized by $\Phi_{\mathrm{lcm}(I)} \cup \Sigma_I$ relatively to the variety of MV-monoids.

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Vietoris endofunctor for closed relations and its de Vries dual*

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Abstract

We generalize the Vietoris endofunctor to the category of compact Hausdorff spaces and closed relations and describe the dual endofunctor on the category of de Vries algebras and subordinations.

Taking the Vietoris hyperspace $\mathbb{V}(X)$ of a compact Hausdorff space X defines an endofunctor $\mathbb{V}$ on the category KHaus of compact Hausdorff spaces and continuous functions. On morphisms, a continuous function $f\colon X \to Y$ is mapped to the function $\mathbb{V}(f)\colon \mathbb{V}(X) \to \mathbb{V}(Y)$ which maps a closed subset F of X to the image $f[F]$ of F under f.

The larger category $\mathsf{KHaus}^{\mathsf{R}}$ of compact Hausdorff spaces and closed relations has been investigated in various works [11, 8, 10, 5, 1]. One appealing feature of $\mathsf{KHaus}^{\mathsf{R}}$ is that it is self-dual. We generalize the Vietoris endofunctor to an endofunctor $\mathbb{V}^{\mathsf{R}}\colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{KHaus}^{\mathsf{R}}$. For a closed relation $R \subseteq X \times Y$, we define $\mathbb{V}^{\mathsf{R}}(R)$ by generalizing the well-known Egli-Milner order: for all closed subsets $F \subseteq X$ and $G \subseteq Y$, we set

$$F \; \mathbb{V}^{\mathsf{R}}(R) \; G \iff G \subseteq R[F] \text{ and } F \subseteq R^{-1}[G],$$

where $R[F]$ is the R-image of F in Y and $R^{-1}[G]$ is the R-preimage of G in X. We show that this defines an endofunctor $\mathbb{V}^{\mathsf{R}}\colon \mathsf{KHaus}^{\mathsf{R}} \to \mathsf{KHaus}^{\mathsf{R}}$ that restricts to the Vietoris endofunctor $\mathbb{V}\colon \mathsf{KHaus} \to \mathsf{KHaus}$ and commutes with the self-duality of $\mathsf{KHaus}^{\mathsf{R}}$.

De Vries duality [7] is a duality for KHaus which associates with each compact Hausdorff space X the boolean algebra $\mathcal{RO}(X)$ of regular opens of X equipped with the proximity relation given by $U \prec V$ iff $\mathsf{cl}(U) \subseteq V$. This yields a duality between KHaus and the category DeV of *de Vries algebras*, i.e. pairs $(B, \prec)$ where B is a complete boolean algebra and $\prec$ is a proximity relation on B. A direct pointfree construction of the endofunctor $\mathsf{DeV} \to \mathsf{DeV}$ dual to $\mathbb{V}\colon \mathsf{KHaus} \to \mathsf{KHaus}$ remained an open problem [4, p. 375]. We resolve this problem as follows.

In [1] we extended de Vries duality to $\mathsf{KHaus}^{\mathsf{R}}$. Let $\mathsf{Stone}^{\mathsf{R}}$ be the full subcategory of $\mathsf{KHaus}^{\mathsf{R}}$ consisting of Stone spaces. Stone duality extends to an equivalence between $\mathsf{Stone}^{\mathsf{R}}$ and the category BA^{S} with boolean algebras as objects and subordination relations as morphisms [6, 9, 1]. This yields an equivalence between $\mathsf{KHaus}^{\mathsf{R}}$ and a category whose objects are pairs (B, S) where B is a boolean algebra and S is a subordination relation on B satisfying axioms generalizing the axioms of an $\mathsf{S5}$-modality. Because of this connection, we termed the pairs (B, S) $\mathsf{S5}$-*subordination algebras* and denoted the resulting category by $\mathsf{SubS5}^{\mathsf{S}}$ [1]. The inclusion $\mathsf{DeV}^{\mathsf{S}} \hookrightarrow \mathsf{SubS5}^{\mathsf{S}}$ of the full subcategory $\mathsf{DeV}^{\mathsf{S}}$ consisting of de Vries algebras

*This presentation is based on [3].
†Presenter

is an equivalence, with quasi-inverse obtained by generalizing the MacNeille completion to $\mathsf{S5}$-subordination algebras [2].

In [12], the endofunctor $\mathbb{K}$ on boolean algebras dual to the Vietoris endofunctor $\mathbb{V}$ on Stone spaces was defined. We lift $\mathbb{K}$ to an endofunctor $\mathbb{K}^{\mathsf{S}}$ on BA^{S} equivalent to $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{Stone}^{\mathsf{R}}$. Finally, we lift $\mathbb{K}^{\mathsf{S}}$ to an endofunctor on $\mathsf{SubS5}^{\mathsf{S}}$ equivalent to $\mathbb{V}^{\mathsf{R}}$ on $\mathsf{KHaus}^{\mathsf{R}}$. Composing it with the MacNeille completion yields an endofunctor on $\mathsf{DeV}^{\mathsf{S}}$ equivalent to $\mathbb{V}^{\mathsf{R}}$. This solves the problem mentioned above in the category $\mathsf{SubS5}^{\mathsf{S}}$, in its full subcategory $\mathsf{DeV}^{\mathsf{S}}$, and finally in DeV via a duality between DeV and a wide subcategory of $\mathsf{DeV}^{\mathsf{S}}$.

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Commutation Groups and State-Independent Contextuality

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Abstract

Contextuality is a key form of non-classicality in quantum mechanics. It has a strong logical content, and contextuality arguments are often referred to as paradoxes. They contradict the basic assumption of classical physics, that observable quantities have well-defined values independently of which measurements are performed.

The strongest form is state-independent contextuality, where the structure of the observables dictates that contextuality arises for any state. The most famous example of this phenomenon is the Peres-Mermin magic square [2], which is constructed from the 2-qubit Pauli group:

$$\begin{array}{ccccc} XI & — & IX & — & XX \\ | & & | & & | \\ IZ & — & ZI & — & ZZ \\ | & & | & & | \\ XZ & — & ZX & — & YY \end{array}$$

Here XI denotes the 2-qubit operator $\sigma_x \otimes I$, and similarly for the other entries. One can now calculate that the operators in each row and column pairwise commute, and hence form a valid measurement context. Moreover, the product of each of the rows, and of the first two columns, is II; while the product of the third column is $-II$. We can use this behaviour to show the impossibility of assigning values to the observables which respects the algebraic structure of commuting observables (*i.e.* those which can be performed together).

We now wish to abstract from the specifics of the Pauli group, and understand the general structure which makes such arguments possible. This leads us to introduce the notion of *commutation group*, to which we now turn.

The idea behind commutation groups is that they are built freely from prescribed commutation relations on a set of generators. Commutation relations play a fundamental role in quantum mechanics, the canonical example being the commutation relation between position and momentum (see e.g. [1]): $[p, q] = i\hbar\mathbb{1}$. We can think of a commutation relation as saying that two elements commute *up to a prescribed scalar*. For this to make sense in a group theoretic context, we need an action of a suitable (classical, hence abelian) group of scalars or "phases" on the group we are constructing. We are interested here in finite group constructions, so we shall work over the finite cyclic groups $\mathbb{Z}_d$, $d \geq 2$.

Given a finite set X of generators, we define a *commutator matrix* to be a map $\mu : X^2 \to \mathbb{Z}_d$ which is skew-symmetric, meaning that $\mu(x, y) = -\mu(y, x)$ for all $x, y \in X$.

We describe the construction of commutation groups from commutator matrices in two ways: by generators and relations, and by a linear algebraic construction. Both are useful, and convey different intuitions. The key relations are the commutation relations $xy \doteq J_{\mu(x,y)}yx$.

Main Results

We summarize the main results:

1. We present commutation groups by generators and relations, parameterised by a commutator matrix. We show that these groups admit a presentation by a confluent and terminating rewriting system, using a given linear order on the generators. The normal forms for this presentation lead to an isomorphism with a form of Heisenberg group [3].

2. We are interested in analyzing contextuality arguments over commutation groups. We use a notion of compatible partial monoid, which allows the idea of closing a set of generators under commuting products to be captured. The scalars embed into the centre of this generated compatible sub-monoid $\mathsf{G}(\mu)$. Non-contextual value assignments correspond to left splittings of this embedding.

3. Contextual words provide witnesses for contextuality, *i.e.* obstructions to the existence of non-contextual value assignments. They generalize the usual argument for the contextuality of Peres-Mermin and other examples in the literature.

   A contextual word is a word over the generators of the commutation group such that:

   - The word can be formed by commuting products from the generators.
   - Each generator occurs a multiple of d times in the word
   - The global phase factor of the word is non-zero.

   The existence of a contextual word implies the contextuality of the commutation group.

4. By a detailed analysis of inversions in words in $\mathsf{G}(\mu)$, we show that contextual words cannot arise in commutation groups over $\mathbb{Z}_d$ for d odd. We explicitly construct non-contextual value assignments for these cases.

5. For even d, we firstly show that if the commutativity graph of $\mathsf{G}(\mu)$ is a cluster graph, then non-contextual value assignments exist. This is shown by verifying the sheaf property for empirical models over the commutation group.

6. In the remaining cases, we give a fine-grained analysis of when contextual words exist. This relies on a reduction of commutator matrices to Darboux normal form, which can be performed even for composite d. Whenever some simple arithmetical criteria are met, contextual words can be constructed over each 4×4 block of this normal form.

7. We show that commutation groups over $\mathbb{Z}_d$ with n generators have faithful representations in the unitary group acting on n qudits. Moreover, the image of these embeddings lies inside the generalized Pauli group over n qudits. Generalized Paulis, which are isomorphic to Heisenberg groups over $\mathbb{Z}_d$, are themselves examples of commutation groups, as are groups associated with Majorana fermions.

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Some Results on Almost Distributive Lattices*

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The concept of an almost distributive lattice (or shortly an ADL) was first introduced by U.M. Swamy and G. C. Rao [2] in 1980 as a common abstraction to most of the existing ring theoretic and lattice theoretic generalization of Boolean algebras. An ADL is an algebra with two binary operations $\vee$ and $\wedge$ which satisfies almost all the properties of a distributive lattice with smallest element 0 except possibly the commutativity of $\vee$, the commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$. It was also observed that any one of these three properties converts an ADL into a distributive lattice.

In this paper, we delve deeper into the study of ADLs and explore some structural properties. We first provide a plenty of examples (finite as well as infinite) supported by Hasse diagrams, exhibiting a variety of ADL properties. In general, it is not known so far whether the $\vee$ operation in ADLs is associative or not. We present a counter example showing that not every ADL is $\vee$-associative. Moreover, we obtain a set of necessary and sufficient conditions for an ADL to be $\vee$-associative. Motivated by this particular example, we obtain a number of subdirectly irreducible finite ADLs other than those given in [2]. But it is still an open problem to prove whether or not there are no more subdirectly irreducible ADLs.

Moreover, we present a number of congruence properties that the class of distributive lattices satisfy but the class of ADLs fails to satisfy. We further state some open problems on finding the sub varieties of the class of ADLs having these congruence properties.

Continuing our investigation on finite ADLs, we obtain an algorithm (or a formula) to determine the cardinality of finite ADLs. Our algorithm is inductive that helps to describe the cardinality of an ADL in terms of the cardinality of a finite distributive lattice (the lattice of its principal ideal); where the cardinality of distributive lattices is given in [1].

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Splittings and finite basis theorems 1: splittings of a lattice

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This is the first part of a two-part talk. We study the splittings in the lattices of quasivarieties, which, as it is very well known, are often the algebraic semantics for finitary structural consequence relations. The splittings in lattices of varieties were extensively studied, and this quest was instigated by seminal paper [2]. In addition, we do not restrict ourselves to complete lattices of all subquasivarieties of a given quasivariety; instead, we often consider an arbitrary complete lattice of quasivarieties of a given type. This, for instance, allows us to study intervals $\{\mathsf{Q}' : \mathbf{Q}(\mathbf{F}_{\mathsf{V}}(\omega)) \subseteq \mathsf{Q}' \subseteq \mathsf{V}\}$, where V is a variety, which sheds light on the properties of the sets of rules admissible in the logic having V as its algebraic semantics. Before dealing with applications it is convenient to lay down some general theory about splittings of a lattice. Let $\mathbf{L}$ be any lattice; an ordered pair (a, b) of elements of $\mathbf{L}$ such that $a \not\leq b$ is a **splitting pair** (**splitting** for short) if for every $c \in \mathbf{L}$, either $a \leq c$, or $c \leq b$. We call a a **splitting element** and b a **co-splitting element**; if (a, b) is a splitting of $\mathbf{L}$ we also will say that the pair (a, b) **splits L**. The concept of splitting pair originated in [3]; there Whitman defined a splitting of a lattice to be a pair (F, I) where F and I are a filter and an ideal of $\mathbf{L}$ respectively, and L is the disjoint union of F and I. Therefore the concept we have introduced is akin to a *principal splitting* in [3]. Given a lattice $\mathbf{L}$ we say that $a \in L$ is **completely join prime** if for all $X \subseteq L$, if $\bigvee X$ exists and $a \leq \bigvee X$, then there is an $x \in X$ with $a \leq x$. A **completely meet prime** element of $\mathbf{L}$ is defined dually. The following facts are either straifghforward or have been shown in [3]:

1. If $(a_1, b_1), (a_2, b_2)$ split $\mathbf{L}$, then $a_1 \leq a_2$ if and only if $b_1 \geq b_2$ and $a_1 < a_2$, if and only if and only if $b_1 > b_2$.
2. If $(a, b), (a, c)$ split $\mathbf{L}$, then $b = c$ and if $(a, c), (b, c)$ split $\mathbf{L}$, then $a = b$. Therefore if a is a splitting element in $\mathbf{L}$ there is a unique co-splitting element a^* (called the **conjugate** of a) such that (a, a^*) splits $\mathbf{L}$; similarly for any co-splitting element b there is a unique splitting element $a \in L$ with $b = a^*$.
3. If a is splitting in $\mathbf{L}$, then a is completely join prime and if $\mathbf{L}$ is complete, then the converse holds.
4. If a is co-splitting in $\mathbf{L}$, then a is completely meet prime and if $\mathbf{L}$ is complete, then the converse holds.
5. Let $\mathbf{M}$ be a complete sublattice of $\mathbf{L}$; if $a \in M$ is splitting in $\mathbf{L}$, then a is splitting in $\mathbf{M}$ with conjugate element $a^* = \bigvee\{b \in M : a \not\leq b\}$.

Antichains of splitting elements are important in a lattice.

Theorem 1. *[1] Let $\mathbf{L}$ be a lattice and S be an infinite antichain of splitting elements. Then,*

1. *$\mathbf{L}$ contains continuum many sublattices;*
2. *$\mathbf{L}$ has infinite ascending and descending chains of elements;*
3. *if $\mathbf{L}$ is complete, then $\mathbf{L}$ is not countable.*

Let $\mathbf{L}$ be a lattice. Element $a \in L$ is **decomposable** if there is a subset $\mathsf{S} \subseteq L$ of completely meet prime elements such that $a = \bigwedge S$; an element $a \in L$ is **join decomposable** if there is a subset $\mathsf{S} \subseteq L$ of completely join prime elements such that $a = \bigvee S$. A decomposition $a = \bigwedge S$ is **irredundant** if for every $b \in S$, $a \neq \bigwedge(S \setminus \{b\})$; a join decomposition is **irredundant** if the dual property holds. Every completely meet prime (join prime) element has a **trivial irredundant decomposition (join decomposition)** consisting of itself. If the lattice $\mathbf{L}$ is complete, then a (join) decomposition is a decomposition into splitting (co-splitting) elements of $\mathbf{L}$.

Proposition 2. *Let $\mathbf{L}$ be a lattice and $S \subseteq \mathbf{L}$ be a set of meet-prime elements. Then the decomposition $a = \bigwedge S$ is irredundant if and only if S is an antichain.*

An element b of lattice $\mathbf{L}$ is **separable** if for every $c \in \mathbf{L}$, if $c \not\leq b$, then there is a splitting element a such that $b \leq a^*$ and $a \leq c$. It follows that the top element of $\mathbf{L}$, if any, is always separable. Dually we say that $c \in L$ is **co-separable** if for every $b \in \mathbf{L}$, if $c \not\leq b$, then there is a splitting element a such that $b \leq a^*$ and $a \leq c$. It follows that the bottom element of $\mathbf{L}$, if any, is always co-separable. A lattice is **separable** if all its elements are separable.

Theorem 3. *[1] For a complete lattice $\mathbf{L}$ the following are equivalent:*

1. *$\mathbf{L}$ is separable;*
2. *every element different from the top is decomposable into a meet of co-splitting elements;*
3. *every element different from the bottom is join decomposable into a join of splitting elements.*

Theorem 4. *[1] Let $\mathbf{L}$ be a complete separable lattice and $\mathsf{S}_\mathbf{L}$ be the set of all its splitting elements. If $\mathsf{S}_\mathbf{L}$ is countable and enjoys the descending chain condition, then the following are equivalent:*

1. *$\mathbf{L}$ is at most countable;*
2. *$\mathsf{S}_\mathbf{L}$ has no infinite antichains;*
3. *each element of $\mathbf{L}$ has a finite irredundant decomposition.*

It is well-known that the class of all quasivarieties and the class of all varieties of algebras of a given type form complete lattices; we are interested in complete sublattices of those lattices. If Q is any quasivariety, then all the subquasivarieties of Q form a complete lattice $\Lambda_q(\mathsf{Q})$. If V is variety then all the subvarieties of V form a complete lattice $\Lambda_v(\mathsf{V})$; since V is also a quasivariety the notation $\Lambda_q(\mathsf{V})$ makes sense. Observe however that $\Lambda_v(\mathsf{Q})$ and $\Lambda_q(\mathsf{V})$ may be quite different; there are examples of varieties whose lattice of subvarieties and whose lattice of subquasivarieties are infinite; there are also examples in which the lattice of subvarieties is countable (even finite!) but the lattice of subquasivarieties is uncountable. Moreover the notation $\Lambda_v(\mathsf{Q})$ for a quasivariety Q also makes sense; however in this case the lattice may not be complete, in that there are examples of quasivarieties Q in which there is no largest variety contained in Q. Now we have set up the playground for applications, that will be dealt with in Part 2.

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Splittings and finite basis theorems
Part II: Complete lattices of subquasivarieties

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This is the second part of a two-part talk, and we use some definitions and notations from the Part I.

For variety V or quasivariety Q, $\Lambda_v(\mathsf{V})$ and $\Lambda_q(\mathsf{Q})$ denote the complete lattices of all subvarieties and all subquasivarieties, of V or Q; as every variety is a quasivariety the notation $\Lambda_q(\mathsf{V})$ also makes sense. The following observation shows the relations between splittings in $\Lambda_v(\mathsf{V})$ and $\Lambda_q(\mathsf{V})$.

Theorem 1. *Let* W *be a variety,* $\mathsf{V} \in \Lambda_v(\mathsf{W})$ *and* $\mathsf{Q} = \mathbf{Q}(\mathbf{F}_{\mathsf{V}}(\omega))$*. Then* V *splits* $\Lambda_v(\mathsf{W})$ *if and only if* Q *splits* $\Lambda_q(\mathsf{W})$*.*

If Q is a quasivariety, algebra $\mathbf{A} \in \mathsf{Q}$ is **Q-irreducible** if there are two elements $a, b \in \mathbf{A}$ such that for any distinct from identity congruence θ of $\mathbf{A}$, if $\mathbf{A}/\theta \in \mathsf{Q}$, then $(a, b) \in \theta$. And $\mathbf{A}$ is finitely Q-presentable if there is is a compact congruence θ of $\mathbf{F}_{\mathsf{Q}}(n)$ such that $\mathbf{A} \cong \mathbf{F}_{\mathsf{Q}}(n)/\theta$.

Similarly to splitting varieties (cf., e.g., [1]) the following holds for splitting quasivarieties.

Theorem 2. *Suppose that* K *is a quasivariety and the pair* $(\mathsf{Q}, \mathsf{Q}^*)$ *splits* $\Lambda_q(\mathsf{K})$*. Then*

1) Q^* *is axiomatized relative to* K *by any quasiequation* φ *such that* $\mathsf{Q}^* \models \varphi$ *and* $\mathsf{Q} \not\models \varphi$*;*
2) Q *is generated by a single finitely generated* Q*-irreducible algebra* $\mathbf{A}$*;*
3) Q *is generated by a single finitely* Q*-presented algebra* $\mathbf{A}$*.*

Among quasiequations mentioned in (1) there always is a **Q-irreducible quasiequation** φ: if $\Phi \models_{\mathsf{Q}} \varphi$, then there is $\varphi' \in \Phi$ such that $\varphi' \models_{\mathsf{Q}} \varphi$; the Q-irreducible quasiequation defining relative to Q the co-splitting subquasivariety is called a **splitting quasiequation**.

The biggest difference between splittings in the lattices of varieties and quasivarieties is that if a pair $(\mathsf{V}, \mathsf{V}^*)$ splits $\Lambda_v(\mathsf{W})$, the V-irreducible algebra generating V is subdirectly irreducible and thus it is W-irreducible. For quasivarieties it is not the case: if pair $(\mathsf{Q}, \mathsf{Q}^*)$ splits $\Lambda_q(\mathsf{K})$, Q may not be generated by any K-irreducible algebras. This observation justifies the following definitions: algebra $\mathbf{A}$ is **self-irreducible** if it is $\mathbf{Q}(\mathbf{A})$-irreducible; algebra $\mathbf{A}$ is a **splitting algebra** in Λ if it is finitely generated self-irreducible and quasivariety $\mathbf{Q}(\mathbf{A})$ splits Λ; and $\mathbf{A}$ is a **strong splitting algebra** if it is a splitting algebra and in addition it is K-irreducible, where K is the top element of Λ. For a quasivariety K by K_{spl} we denote the class of all algebras splitting $\Lambda_q(\mathsf{K})$. On K_{spl} we also define a quasi-order by letting for any $\mathbf{A}, \mathbf{B} \in \mathsf{K}_{spl}$, $\mathbf{A} \leq \mathbf{B} \leftrightharpoons \mathbf{Q}(\mathbf{A}) \subseteq \mathbf{Q}(\mathbf{B})$; and this quasi-order can be easily converted into a partial order on the cosets.

The notion of separability was defined in the Part I. For instance, if quasivariety Q and all its subquasivarieties have the finite embeddability property (FEP for short), that is if each quasivariety from $\Lambda_q(\mathsf{Q})$ is generated by its finite members, then $\Lambda_q(\mathsf{Q})$ is separable.

Theorem 3. *Let* Λ *be a complete lattice of quasivarieties and* K *be its top element. If* $\mathsf{Q} \in \Lambda$ *is separable, then it has a basis consisting of splitting quasiequations relative to* K*. Thus, if* Λ *is separable, then every member of* Λ *has a basis relative to* K *consisting of splitting quasiequations.*

A quasivariety Q is **primitive** if every its subquasivariety can be defined relative to Q by a set of identities, i.e. for every $\mathsf{Q}' \in \Lambda_q(\mathsf{Q})$, $\mathsf{Q}' = \mathsf{Q} \cap \mathbf{V}(\mathsf{Q}')$, where $\mathbf{V}(\mathsf{Q}')$ is the variety generated by Q'. The primitive quasivarieties are the algebraic counterparts of hereditarily structurally complete finitary structural consequence relations. And Q is **weakly primitive** if in every $\mathsf{Q}' \in \Lambda_q(\mathsf{Q})$, every algebra $\mathbf{A} \in \mathsf{Q}'$ is a subdirect product of Q-irreducible algebras from Q'.

Theorem 4. *Every primitive quasivariety is weakly primitive. Moreover, quasivariety Q is weakly primitive if and only if every self-irreducible algebra in Q is Q-irreducible.*

A quasivariety Q is **weakly tame** if every finitely generated Q-irreducible algebra in Q is Q-splitting (and thus it is strong Q-splitting). For instance, every quasivariety of finite type with the FEP (hence any locally finite quasivariety of finite type) is weakly tame.

Corollary 5. *If Q is weakly tame and weakly primitive, then $\mathsf{Q} = \mathbf{Q}(\mathsf{Q}_{spl})$.*

If $\mathsf{Q}' \subseteq \mathsf{Q}$, we define $I[\mathsf{Q}', \mathsf{Q}] = \{\mathsf{Q}'' : \mathsf{Q}' \subseteq \mathsf{Q}'' \subseteq \mathsf{Q}\}$.

Theorem 6. *Let Q be weakly primitive, weakly tame quasivariety of finite type and $\mathsf{Q}' \subseteq \mathsf{Q}$ such that every quasivariety in $I[\mathsf{Q}', \mathsf{Q}]$ has the FEP. Then the following are equivalent:*

1) *every $\mathsf{Q}'' \in I[\mathsf{Q}', \mathsf{Q}]$ has a finite basis relative to Q;*
2) *$I[\mathsf{Q}', \mathsf{Q}]$ is countable;*
3) *$\mathsf{Q}_{spl} \setminus \mathsf{Q}'_{spl}$ has no infinite antichain;*
4) *$I[\mathsf{Q}', \mathsf{Q}]$ enjoys the descending chain condition.*

Corollary 7. *If Q is weakly primitive, of finite type and finitely generated then $\Lambda_q(\mathsf{Q})$ is finite and all its subquasivarieties have a finite basis relative to Q.*

Proof. The proof follows from the observation that Q has just a finite (up to isomorphism) set of strong Q-splitting algebras. □

Corollary 8. *If Q is primitive, finitely axiomatizable and of finite type, then every finitely generated subquasivariety of Q is finitely axiomatizable.*

Remark 9. Corollary 7 can be seen as a version of Baker's Finite Basis Theorem for quasivarieties; our version differs from the one in [2], in that we drop relative congruence distributivity and add weak primitivity.

Note that primitivity is essential in Corollary 7. In [3] (also see [4, Section 4.5]) Rybakov gave an example of finite Heyting algebra $\mathbf{A}$ with $\mathbf{Q}(\mathbf{A})$ not having a finite basis relative to variety of all Heyting algebras and therefore, relative to $\mathbf{V}(\mathbf{A})$. We note that $\Lambda_q(\mathbf{V}(\mathbf{A}))$ is infinite, while $\Lambda_v(\mathbf{V}(\mathbf{A}))$ is finite. Rybakov's example also shows that congruence distributive varieties may have subquasivarieties which are not relatively congruence distributive.

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Structurally complete finitary extensions of positive Łukasiewicz logic

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A large class of substructural logics is given by the finitary extensions of the *Full Lambek Calculus* $\mathcal{FL}$ (see [4]). It is well known that all these finitary extensions are algebraizable with an *equivalent algebraic semantics* (in the sense of Blok-Pigozzi [3]) that is at least a quasivariety of algebras. An important subfamily of substructural logics over $\mathcal{FL}$ consists of logics that satisfy both exchange and weakening but lack contraction. In this context, one of the most studied examples is Łukasiewicz's infinite-valued logic, which we denote by $\mathcal{MV}$ [6], and its algebraic counterpart given by the variety of MV-algebras. Since all extensions of $\mathcal{FL}$ have a primitive connective $\mathbf{0}$ that denotes the *falsum*, if $\mathcal{L}$ is an extension of $\mathcal{FL}$, it makes sense to study its *positive* fragment $\mathcal{L}^+$ (i.e. the logic obtained from $\mathcal{L}$ by removing $\mathbf{0}$ from the signature), which is still algebraizable.

In this framework, relevant algebraic structures are given by *hoops*, a particular variety of residuated monoids which were defined in an unpublished manuscript by Büchi and Owens. As a most relevant subvariety, the variety WH of *Wajsberg hoops*, is the equivalent algebraic semantics of $\mathcal{MV}^+$, i.e. the positive fragment of Łukasiewicz logic. Its relevance extends also to the purely algebraic framework, since Wajsberg hoops can be used to describe subdirectly irreducible hoops and the whole variety of hoops can be obtained as the join of iterated powers of the variety of WH, in the sense defined in [2]. In this contribution, we study structural completeness in sub(quasi)varieties of Wajsberg hoops.

A rule is *admissible* in a logic if, when added to its calculus, it does not produce new theorems. A logic $\mathcal{L}$ is *structurally complete* if every admissible rule of $\mathcal{L}$ is derivable in $\mathcal{L}$; it is *hereditarily structurally complete* if every finitary extension of $\mathcal{L}$ is structurally complete. In an algebraizable logic, these notions correspond to the associated quasivariety being *structural* and *primitive* respectively. A quasivariety Q is *structural* if for every subquasivariety $\mathsf{Q}' \subseteq \mathsf{Q}$, $\mathbf{H}(\mathsf{Q}') = \mathbf{H}(\mathsf{Q})$ implies $\mathsf{Q}' = \mathsf{Q}$ (where $\mathbf{H}(\mathsf{Q})$ is the class of all homomorphic images of algebras in Q); moreover we define the *structural core of* Q as the smallest $\mathsf{Q}' \subseteq \mathsf{Q}$ such that $\mathbf{H}(\mathsf{Q}) = \mathbf{H}(\mathsf{Q}')$. A quasivariety Q is *primitive* if every subquasivariety of Q is structural.

Notice that a quasivariety Q is structural if and only if it coincides with its structural core. As a consequence the structural subquasivarieties of a quasivariety Q are exactly those that coincide with the structural cores of Q' for some $\mathsf{Q}' \subseteq \mathsf{Q}$; even more, since $\mathbf{H}(\mathsf{Q})$ is a variety, the structural subquasivarieties of a variety V are exactly the structural cores of V' for some subvariety V' of V.

In this work, we characterize all the structurally complete finitary extensions of $\mathcal{MV}^+$. Moreover, we provide some examples of finitary extensions of $\mathcal{MV}^+$ that are hereditarily structurally complete and others that are not. Algebraically, this corresponds to classifying all the structural quasivarieties and studying some of the primitive quasivarieties of Wajsberg hoops.

Starting from the work of Gispert about MV-algebras ([5]), we begin our analysis from varieties, where the results are almost straightforward. It is known ([1]) that every proper variety of Wajsberg hoops is generated by a finite number of chains of the type $\mathbf{Ł}_n, \mathbf{Ł}_n^\infty$ or $\mathbf{C}_\omega$, where $\mathbf{Ł}_n = \Gamma(\mathbb{Z}, n)$, $\mathbf{Ł}_n^\infty = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (n, 0))$ and $\mathbf{C}_\omega$ is the negative cone of $\mathbb{Z}$ with the operations defined in the obvious way (the notation for these constructions is the one of Mundici [7]). To be more precise, every proper subvariety of Wajsberg hoops can be associated with a particular kind of triple (I, J, K), called *reduced triple*, where I, J are finite subsets of $\mathbb{N} \setminus \{0\}$, $K \subseteq \{\omega\}$ and if $J \neq \emptyset$ then $K = \emptyset$. The connection is the following: if $P = (I, J, \emptyset)$ then $\mathsf{V}(P) = \mathbf{V}(\{\mathbf{Ł}_i : i \in I\} \cup \{\mathbf{Ł}_j^\infty : j \in J\})$ (the variety generated by all $\mathbf{Ł}_i$ and $\mathbf{Ł}_j^\infty$), if $P = (I, \emptyset, \{\omega\})$ then $\mathsf{V}(P) = \mathbf{V}(\{\mathbf{Ł}_i : i \in I\} \cup \{\mathbf{C}_\omega\})$.

Theorem. *Let $\mathsf{V} = \mathsf{V}(I, J, K)$ be a proper subvariety of Wajsberg hoops. Then V is structural if and only if either $J = \emptyset$, or $J = \{1\}$.*

Notice that, if $\mathsf{V} = \mathsf{V}(I, J, K)$ with either $J = \emptyset$ or $J = \{1\}$, every subquasivariety of V is still a variety of that form, so as an immediate consequence we get that a variety of Wajsberg hoops is structural if and only if it is primitive.

Moving on to quasivarieties, the characterization becomes more difficult to prove, but the results are still quite simple to present. Given a reduced triple, we define $\mathsf{Q}[I, J, \emptyset] = \mathbf{Q}(\{\mathbf{Ł}_i : i \in I\} \cup \{\mathbf{Ł}_{j,1} : j \in J\})$ (the quasivariety generated by all $\mathbf{Ł}_i$ and $\mathbf{Ł}_{j,1} = \Gamma(\mathbb{Z} \times_l \mathbb{Z}, (j, 1))$ [7]) and $\mathsf{Q}[I, \emptyset, \{\omega\}] = \mathbf{Q}(\{\mathbf{Ł}_i : i \in I\} \cup \{\mathbf{C}_\omega\})$. Using this notation, we can characterize all the structural quasivarieties of Wajsberg hoops.

Theorem. *Let Q be a quasivariety of Wajsberg hoops. Then Q is structural if and only if either Q is the structural core of WH or $\mathsf{Q} = \mathsf{Q}[I, J, K]$ for some reduced triple (I, J, K).*

Unlike varieties, we fall short of characterizing all primitive quasivarieties, due to the lack of understanding of the lattice of all the subquasivarieties of Wajsberg hoops. Despite that, we managed to find some examples of primitive and non-primitive quasivarieties.

Proposition. *Let $\mathsf{Q} = \mathsf{Q}[I, J, \emptyset]$ with $(I, J, \emptyset)$ reduced triple; if there exists $m \neq 1$ and there exist $i \in I, j \in J$ such that m divides i and j, then Q is not primitive. On the other hand, if $\mathsf{Q} = \mathsf{Q}[\emptyset, \{p\}, \emptyset]$ where p is a prime number, then Q is primitive.*

In particular, these examples show that in Wajsberg hoops there exist nontrivial primitive proper quasivarieties, but not all structural quasivarieties are primitive.

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Completeness of the GL.3 provability logic for the intersection of normal measures

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Abstract

Provability logic GL is known to be sound and complete w.r.t. *scattered topological spaces*, namely spaces (X, τ) where every $A \subset X$ has an isolated point. The topological interpretation is the following: topological model is a pair $\langle (X, \tau), v \rangle$, where (X, τ) is a topological space and $v : \mathsf{Vars} \to PX$, which yields an interpretation:

- $[\![\top]\!] = X; [\![\bot]\!] = \emptyset; [\![p]\!] = v(p)$;
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$;
- $[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!]$;
- $[\![\Diamond\varphi]\!] = d_\tau [\![\varphi]\!]$;

where $d_\tau A = \{x : \forall U \in \tau_i \exists y \neq x (y \in U \cap A)\}$ for each $A \subset X$, we call d_τ the derivative operator.

One can show that GL is the logic of the class of scattered spaces. Moreover, GL is known to be Kripke complete w.r.t. to finite irreflexive trees [6], hence by regarding Kripke models as topologies generated by the upsets, we are already getting topological completeness for the class of scattered spaces. Moreover, GL is topologically complete with respect to any single ordinal $\geq \omega^\omega$:

Theorem 1 (Abashidze[1], Blass[2]). *Consider an ordinal $\Omega \geq \omega^\omega$ with its order topology, then $Log(\Omega) = \mathsf{GL}$.*

There is a number of other peculiar yet natural examples of scattered spaces. Most of them arise in the context of set theory, so expectedly the question of completeness is independent for some of them. They are worth studying, for we can see how modal logic can reflect properties of somewhat complicated objects. Another important motivation behind this is that GLP – the polymodal generalization of GL – is Kripke incomplete and the candidates for the right topological models dwell in the realm of topologies on ordinals, hence related questions naturally emerge in the study of GLP.

A. Blass in [2] provided a characterization of topologies on Ord for which GL is sound. Although his results were formulated in terms of sequences of filters, filters and topologies can be mutually interpreted, by regarding filter on α as the set of punctured neighborhoods of α. Some natural topologies that satisfy these conditions were mentioned by A. Blass in the same

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article, namely topologies corresponding to: the end-segment filter, the club-filter, the subtle filter, the ineffable filter and the filter

$$M_\kappa = \bigcap\{U : U \text{ is a normal measure on } \kappa\}$$

For this filter we call the corresponding topology τ_U, the topology given by letting $A \subset \kappa$ be a punctured neighborhood of κ if $A \in M_\kappa$ (as well as τ'_U which is essentially the same, but relativized to pseudonormal filters). These topologies are the main subject of our study.

Golshani and Zoghifard in [3] have shown that there is a model of ZFC, where $\mathrm{Log}(\langle \mathsf{Ord}, \tau_U \rangle) = \mathsf{GL}$, provided there exists infinitely many strong cardinals. Whereas Blass has mentioned that GL can fail to be the logic of this topology, since there is a model $L[\mathcal{U}]$ – class of sets constructible from a sequence of normal ultrafilters (cf.[5], [4]). In this model a non-theorem of GL holds for τ_U, however it was open what exactly the logic of it is. Let

$$\mathsf{GL.3} = \mathsf{GL} + \Box(\Box A \to B) \vee \Box(\Box B \wedge B \to A)$$

The main result of our investigation is:

Theorem 2. $\mathrm{Log}(\langle \mathsf{Ord}, \tau \rangle) = \mathsf{GL.3}$ *if* $V = L[\mathcal{U}]$ + *"there exists a measurable cardinal of Mitchell order* n *for each* $n < \omega$*". Where* $\tau \in \{\tau_U, \tau'_U\}$.

Moreover, we have shown

Theorem 3. $\mathrm{Log}(\langle \mathsf{Ord}, \tau'_U \rangle) = \mathsf{GL.3}$ *if* AD *holds.*

Generalization of this result is the matter of our future research.

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Colimits of Heyting Algebras through Esakia Duality

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In contrast to Boolean algebras and Distributive lattices, the variety of Heyting algebras is not locally finite, and in fact, none of its free finitely generated algebras are finite. The associated difficulty of understanding the free algebras has motivated a wealth of research into describing such algebras. Such investigations were carried out by Bellissima [3], Grigolia [6], [7] and Urquhart [8], as well as later by Ghilardi [5]. These have allowed semantic proofs of several key facts regarding this category of algebras: the fact that finitely presented Heyting algebras are bi-Heyting algebras being a prime example. Despite this wealth of work, a description of the free Heyting algebra on any number of generators seems to not have been presented in the literature. In this talk (based on an available preprint [1]) we generalize Ghilardi's [5] construction of the free Heyting algebra generated by a finite distributive lattice to any distributive lattice. The key technical tool employed is Priestley duality, as well as the following adaptations of Ghilardi's construction:

Definition 1. Let X, Y, Z be Priestley spaces, and $g : X \to Y$ and $f : Y \to Z$ be Priestley morphisms. We say that f is *open relative to* g (g-open for short) if it satisfies the following:

$$\forall a \in X, \forall b \in Y, (f(a) \leq b \implies \exists a' \in X, (a \leq a' \ \& \ g(f(a')) = g(b)). \tag{*}$$

Given $S \subseteq X$ a closed subset, we say that S is *g-open* (understood as a poset with the restricted partial order relation) if the inclusion is itself g-open.

Definition 2. Let $g : X \to Y$ be a map between Priestley spaces. Then consider

$$V_g(X) := \{C \subseteq X : C \text{ is closed, rooted and } g\text{-open } \},$$

with the topology given by a subbasis consisting of sets of the form

$$[U] = \{C \in V_g(X) : C \subseteq U\} \text{ and } \langle V \rangle = \{C \in V_g(X) : C \cap V \neq \emptyset\},$$

where U, V are clopen subsets of X.

The following can then be shown:

Proposition 3. *Given $g : X \to Y$ a Priestley morphism, the space $(V_g(X), \preceq)$ is a Priestley space, equipped with a continuous surjection $r_g : V_g(X) \to X$ sending each rooted subset to its root.*

The construction V_g enjoys a specific universal property:

Lemma 4. Given a Priestley map $g : X \to Y$, the construction V_g enjoys the following property: given a Priestley space Z with a g-open continuous and order-preserving map $h : Z \to X$, there exists a unique r_g-open, continuous and order-preserving map h' such that the triangle in Figure 1 commutes.

$$\begin{tikzcd} Z \arrow[rr, "h'"] \arrow[dr, "h"'] & & V_g(X) \arrow[dl, "r"] \\ & X & \end{tikzcd}$$

Figure 1: Commuting Triangle of Priestley spaces

Definition 5. Let $g : X \to Y$ be a Priestley morphism. The *g-Vietoris complex* over X $(V^g_\bullet(X), \leq_\bullet)$, is a sequence

$$(V_0(X), V_1(X), ..., V_n(X))$$

connected by morphisms $r_i : V_{i+1}(X) \to V_i(X)$ such that:

1. $V_0(X) = X$;
2. $r_0 = g$
3. For $i \geq 0$, $V_{i+1}(X) := V_{r_i}(V_i(X))$;
4. $r_{i+1} = r_{r_i} : V_{i+1}(X) \to V_i(X)$ is the root map.

We denote the projective limit of this family by $V_G^g(X)$, and omit it when g is the terminal map to $\mathbf{1}$.

Theorem 6. *The assignment V_G is a functor mapping the category* **Pries** *of Priestley spaces and Priestley morphisms to the category* **Esa** *of Esakia spaces; indeed it is the right adjoint of the inclusion.*

As applications, we obtain new proofs of old results, as well as some new facts: (1) a description of free Heyting algebras on any number of generators is given; (2) a description of coproducts of Heyting algebras is given, and it is shown that the category of Heyting algebras is co-distributive; (3) A description of pushouts of Heyting algebras is given, and it is shown directly that the coprojections of Heyting algebras to the pushout are injective (yielding, as a corollary, the amalgamation property).

We also consider two generalizations of these results:

1. We consider the construction obtained when restricting to specific subvarieties of **HA**, such as KC-algebras and LC-algebras (often called "Gödel algebras"), and show that adaptations of the above ideas yield descriptions of the free algebras in these varieties.
2. We study the category of image-finite posets and p-morphisms and its relationship to the category of posets. We show that a similar adjunction holds here. This connects with recent work by de Berardinis and Ghilardi [4], and provides a generalization of the n-universal model for arbitrary finite posets.

We also highlight some connections to coalgebraic representations of intuitionistic modal logic, which we investigate in depth in a paper with Nick Bezhanishvili [2]. We conclude by pointing some further avenues of exploration, as well as some questions left open by the above research:

Problem 7. Is $V_g(X)$ an Esakia (bi-Esakia) space whenever X is an Esakia (bi-Esakia) space?

Problem 8. Does the inclusion of $\mathbf{Pos}_p$, the category of posets with p-morphisms, into **Pos** admit a right adjoint?

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Strict Leibniz Hierarchy and Categories of Logics

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The Leibniz hierarchy is a classification system for propositional logics in terms of their behavior with respect to the Leibniz congruence/operator Ω. In [AP], we propose the introduction of two new classes of logics into the Leibniz hierarchy: the class of Ω-natural logics and its intersection with the class of truth-equational logics, which we called truth-natural. In short, we say that a logic has the property of Ω-naturality when its Leibniz operator commutes with inverse substitutions (endomorphisms on the algebra of formulas), or equivalently, inverse arbitrary homomorphisms. It is then clear that the well-known classes of equivalential logics and algebraizable logics, for instance, represent the intersection between the class of Ω-natural logics and the classes of protoalgebraic and weakly algebraizable logics, respectively.

In order to substantiate the relevance of these new classes, we looked to the attempted formalization of the Leibniz hierarchy conducted in [JM1, JM2, JM3], which also discusses the collection of all logics viewed as a poset whose partial order is given by existence of interpretations (in a specific sense) between logics, drawing inspiration from the Maltsev hierarchy of Universal Algebra. Furthermore, the question of whether there is a precise relation between the so-called Leibniz classes and the behavior of the Leibniz operator is raised in [JM2, Prob. 2].

We show in [AP, Prop. 3.3, Thm. 4.14] not only a sufficient condition for Ω-naturality determined solely by the underlying language of a given logic, but also that the class of Ω-natural logics do not, in fact, comprise a Leibniz class [JM2, Thm. 2.2 (ii)]. In this case, we have a negative answer to the aforementioned open problem associating Leibniz classes and the Leibniz operator.

With this issue in mind, we will opt to distinguish the definition of Leibniz-reduced interpretation (LR-interpretation, for short) from that of Suszko-reduced interpretation (SR-interpretation, for short):

Definition. *(translation, SR-interpretation, LR-interpretation)*

- *Given two languages Σ and Σ', a* **translation** *[JM1, Def. 3.1] from Σ to Σ' is any arity-preserving map $\tau : \Sigma \to Fm_{\Sigma'}$. We can see that any translation τ induces a contravariant functor $\tau^* : \Sigma'{-}Str \to \Sigma{-}Str$.*

- *Given a Σ-logic S and a Σ'-logic S', an* **SR-interpretation** *[JM1, Def. 3.2] of S into S' is a translation τ from Σ to Σ' such that $\tau^*[Mod^{\Xi}(S')] \subseteq Mod^{\Xi}(S)$.*

- *Given a Σ-logic S and a Σ'-logic S', an* **LR-interpretation** *of S into S' is a translation τ from Σ to Σ' such that $\tau^*[Mod^*(S')] \subseteq Mod^*(S)$,*

where Mod^{Ξ} (resp. Mod^*) denotes the class of all matrix models for a given logic whose filters have Suszko (resp. Leibniz) congruences coinciding with the identity relation, which are called ***Suszko-reduced*** *(resp.* ***Leibniz-reduced****, or simply* ***reduced****) matrices.*

It is therefore easy to see that any LR-interpretation is also an SR-interpretation, since the Suszko congruence is always contained in the Leibniz congruence. This observation, together with the fact [JM1, Prop. 3.3] that SR-interpretations are also flexible morphisms of logics in the sense of [AFLM], shows that the following inclusion of (wide) subcategories holds: $Log_{LR} \hookrightarrow Log_{SR} \hookrightarrow Log_f$, which denote the categories of all logics with LR-interpretations, SR-interpretations and flexible morphisms, respectively. In fact, Log_{SR} is simply the categorical reframing of the poset of all logics Log as defined in [JM1, Def. 3.5] before passing through the quotient of equi-interpretability. In this sense, we can then investigate the relationships between those three categories, e.g. finding possible equivalences, adjoints, their associated monads and algebras.

Moreover, we would also like to point out that the distinction between the Suszko and Leibniz congruences (and therefore between SR-interpretations and LR-interpretations) is non-existent among protoalgebraic logics [Cze, Thm. 1.5.4]. In Czelakowsi's words, "the list of plausible properties of the Suszko operator, parallel to those of the Leibniz one, may thus serve as a basis for distinguishing a hierarchy of all logics which, when restricted to protoalgebraic logics, agrees with the [Leibniz] hierarchy" [Cze, p. 9].

Now, in the context of Leibniz conditions and classes (see [JM2, Def. 2.1], if we replace (i.e. strengthen) SR-interpretability with LR-interpretability in the corresponding definitions, we can then define what we shall call **strict Leibniz conditions**, **strict Leibniz classes** and the **strict Leibniz hierarchy**. Therefore, once we make the appropriate tweaks to the surrounding concepts, this adaptation also preserves most, if not all properties analogous to those of Leibniz classes (such as the appealing [JM2, Thm. 2.2]). Even so, it remains to be verified whether Ω-natural (resp. truth-natural) logics do indeed form a strict Leibniz class via this characterization.

In conclusion, we also leave as a possibility for further investigation that of determining which categorical properties Log_{SR} and Log_{LR} possess. It is already known that Log_{SR} admits weak products, but not necessarily even finite weak coproducts, given that the poset Log admits arbitrary infima but not necessarily even finite suprema (see [JM1, Thm. 4.6, 5.1]). In addition, both Log_{SR} and Log_{LR} seem to be very good candidates for being factorization systems, considering the decomposition [JM1, Prop. 3.8] of any SR-interpretation into a compatible expansion and a term-equivalence.

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On continuity and openness of maps between locales

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Let $f\colon X \to Y$ be a function between topological spaces (or, more generally, closure spaces) and define the pair of assignments

$$\begin{tikzcd} \mathrm{Closed}(X) \arrow[r, bend left, "f^{\rightarrow}"] & \mathrm{Closed}(Y) \arrow[l, bend left, "f^{\leftarrow}"] \end{tikzcd} \qquad \begin{array}{rcl} A & \mapsto & \mathrm{cl}\,(f[A]) \\ \mathrm{cl}\,(f^{-1}[B]) & \leftarrow\!\shortmid & B \end{array}$$

It is very easy to check that

f is continuous iff $(f^{\rightarrow}, f^{\leftarrow})$ is an adjoint pair.

This idea that the continuous maps between topological spaces (and, more generally, closure spaces) correspond to certain adjoint pairs of maps between the involved closure systems, by assigning with any continuous map the lifted map taking the closures of images as left adjoint and the preimage map as right adjoint, is well-known [2, 3, 7]. This idea was also explored in the pointfree setting in the forerunner article [4]. Our goal in this talk is to revisit those adjunctions and to present new characterizations of *continuous* (that is, *localic*) maps and *open* maps (that is, plain maps with *open images of open sublocales* [5, 6]) between locales, in terms of certain Galois adjunctions between the locales of open sublocales or between the colocales of closed sublocales. With these results we can better understand the differences between the morphisms in the classical and pointfree settings.

Let **Loc** denote the category of locales and localic maps ([8]). Recall that a map $f\colon L \to M$ between locales L and M is a *localic map* (the counterpart of a *continuous map* in the pointfree setting) if

(L1) it preserves arbitrary meets (and hence it has a left adjoint $h\colon M \to L$),

(L2) $f(a) = 1 \;\Rightarrow\; a = 1$, and

(L3) $f(h(a) \to b) = a \to f(b)$ for every $a \in M$, $b \in L$.

In a locale L and any $a \in L$, we consider the *open* and *closed* sublocales

$$\mathfrak{o}(a) = \{x \mid a \to x = x\} = \{a \to x \mid x \in L\} \quad \text{and} \quad \mathfrak{c}(a) = {\uparrow} a = \{x \in L \mid x \geq a\}$$

and denote by $\mathfrak{o}L$ and $\mathfrak{c}L$ respectively the sets of open and closed sublocales of L. We start by generalizing the closure and interior operators from sublocales to general subsets and with a discussion of the problems that may emerge from doing so. Then, given a plain map $f\colon L \to M$, we consider $f^*\colon M \to L$ and $f_!\colon L \to M$, given by $f^*(b) = \bigvee\{a \in L \mid f[\mathfrak{o}(a)] \subseteq \mathfrak{o}(b)\}$ and $f_!(a) = \bigvee\{b \in M \mid \mathfrak{o}(b) \subseteq f[\mathfrak{o}(a)]\}$, and the remaining maps in the following diagram, defined by

$$f_{\mathfrak{o}}^{\rightarrow}(\mathfrak{o}(a)) = \neg(\mathrm{cl}\,(f[\mathfrak{c}(a)])), \quad f_{\mathfrak{o}}^{\leftarrow}(\mathfrak{o}(b)) = \mathrm{int}\,(f^{-1}[\mathfrak{o}(b)]), \quad f_{\mathfrak{o}}^{\Rightarrow}(\mathfrak{o}(a)) = \mathrm{int}\,(f[\mathfrak{o}(a)]),$$

$$f_{\mathfrak{c}}^{\rightarrow}(\mathfrak{c}(a)) = \mathrm{cl}\,(f[\mathfrak{c}(a)]), \quad f_{\mathfrak{c}}^{\leftarrow}(\mathfrak{c}(b)) = \mathrm{cl}\,(f^{-1}[\mathfrak{c}(b)]), \quad f_{\mathfrak{c}}^{\Rightarrow}(\mathfrak{c}(a)) = \neg(\mathrm{int}\,(f[\mathfrak{o}(a)])).$$

*Joint work with Jorge Picado [1].

$$
\begin{tikzcd}
\mathfrak{o}L \arrow[r, "f_{\mathfrak{o}}^{\Rightarrow}"] \arrow[dd, bend left, "\neg"] & \mathfrak{o}M \arrow[r, "f_{\mathfrak{o}}^{\leftarrow}"] \arrow[dd, bend left, "\neg"] & \mathfrak{o}L \arrow[r, "f_{\mathfrak{o}}^{\rightarrow}"] \arrow[dd, bend left, "\neg"] & \mathfrak{o}M \arrow[dd, bend left, "\neg"] \\
L \arrow[u, "\mathfrak{o}"'] \arrow[d, "\mathfrak{c}"] \arrow[r, "f_!"] & M \arrow[u, "\mathfrak{o}"'] \arrow[d, "\mathfrak{c}"] \arrow[r, "f^*"] & L \arrow[u, "\mathfrak{o}"'] \arrow[d, "\mathfrak{c}"] \arrow[r, "f"] & M \arrow[u, "\mathfrak{o}"'] \arrow[d, "\mathfrak{c}"] \\
\mathfrak{c}L \arrow[r, "f_{\mathfrak{c}}^{\Rightarrow}"] \arrow[uu, bend left, "\neg"] & \mathfrak{c}M \arrow[r, "f_{\mathfrak{c}}^{\leftarrow}"] \arrow[uu, bend left, "\neg"] & \mathfrak{c}L \arrow[r, "f_{\mathfrak{c}}^{\rightarrow}"] \arrow[uu, bend left, "\neg"] & \mathfrak{c}M \arrow[uu, bend left, "\neg"]
\end{tikzcd}
$$

One has:

1. The pair $(f_{\mathfrak{c}}^{\rightarrow}, f_{\mathfrak{c}}^{\leftarrow})$ is an adjoint pair if and only if f is meet-preserving.
2. If f is order-preserving then the pair $(f_{\mathfrak{o}}^{\leftarrow}, f_{\mathfrak{o}}^{\rightarrow})$ is an adjoint pair if and only if f is a localic map.
3. If f is meet-preserving then:
   (a) The pair $(f_{\mathfrak{o}}^{\Rightarrow}, f_{\mathfrak{o}}^{\leftarrow})$ is an adjunction if and only if f is open.
   (b) The pair $(f_{\mathfrak{c}}^{\leftarrow}, f_{\mathfrak{c}}^{\Rightarrow})$ is an adjunction if and only if f is an open localic map.

An attractive feature of these adjunctions is that they are all concerned with elementary ideas and basic concepts of localic topology: the use of the concrete language of sublocales and its technique simplifies the reasoning. Taking advantage of the generalization of the interior operator and of the characterization of localic maps in [4], we further obtain the following results:

Proposition 1. *A plain map $f\colon L \to M$ is a localic map if and only if*

$$\neg(\operatorname{int}(f^{-1}[\mathfrak{o}(b)])) = f^{-1}[\mathfrak{c}(b)] \quad \text{for every } b \in M.$$

Proposition 2. *A plain map $f\colon L \to M$ is an open localic map if and only if*

$$\neg(\operatorname{int}(f^{-1}[T])) = f^{-1}[\neg(\operatorname{int} T)] \quad \text{for every sublocale } T \text{ of } M.$$

If time permits we will also refer to some open questions.

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Weak distributive laws between powerspaces over stably compact spaces

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In the study of programming languages, effects may be modeled using monads. For instance, the powerset monad $\mathcal{P}$ on the category **Set** models non-determinism: a non-deterministic function from a set X to a set Y can be seen as a (deterministic) function $X \to \mathcal{P}Y$. Similarly, the distribution monad $\mathcal{D}$ and the corresponding functions $X \to \mathcal{D}Y$ model probabilistic non-determinism: the monad $\mathcal{D}$ sends a set X to the set of finitely supported probability distributions on X. Ordinary and probabilistic non-determinism may be combined in **Set** using *weak distributive laws*, and in this work we study how to combine these effects in a topological setting: we construct weak distributive laws between powerspace monads over the category of stably compact spaces, with the hope that weak distributive laws involving probabilistic powerspace monads may also be constructed in future work.

Given two monads $\mathcal{S}$ and $\mathcal{T}$ on a category $\mathcal{C}$ corresponding to two effects, the combination of these effects may be modeled by composing the monads $\mathcal{S}$ and $\mathcal{T}$. This can be done by finding a *distributive law* between the two monads, i.e. a natural transformation $\mathcal{TS} \Rightarrow \mathcal{ST}$ satisfying certain axioms. Such a distributive law yields a new monad $\mathcal{S} \circ \mathcal{T}$ on $\mathcal{C}$ whose underlying functor is the composite of the functors underlying $\mathcal{S}$ and $\mathcal{T}$.

But such distributive laws may not necessarily be unique and may not exist. In particular, there is no distributive law $\mathcal{PP} \Rightarrow \mathcal{PP}$ nor $\mathcal{DP} \Rightarrow \mathcal{PD}$ for combining two layers of non-determinism or non-determinism with probabilistic non-determinisim [9, 1]. Still, there are natural transformations $\lambda^{\mathcal{P}/\mathcal{P}} : \mathcal{PP} \Rightarrow \mathcal{PP}$ and $\lambda^{\mathcal{D}/\mathcal{P}} : \mathcal{DP} \Rightarrow \mathcal{PD}$ that satisfy all but one of the axioms required for them to be distributive laws [2, 6]: these natural transformations are called *weak distributive laws*. Again, weak distributive laws $\mathcal{TS} \Rightarrow \mathcal{ST}$ yield a composite monad, but its underlying functor need not be the composite of the functors underlying $\mathcal{S}$ and $\mathcal{T}$ anymore.

The powerset monad has a topological analogue, the Vietoris monad $\mathcal{V}$ on the category of compact Hausdorff spaces (it sends a compact Hausdorff space to the space of its closed subsets equipped with the Vietoris topology). It was shown recently that the weak distributive law $\lambda^{\mathcal{P}/\mathcal{P}}$ also has a topological analogue $\lambda^{\mathcal{V}/\mathcal{V}} : \mathcal{VV} \Rightarrow \mathcal{VV}$ [7].

What about a topological analogue of $\lambda^{\mathcal{D}/\mathcal{P}}$? Goy conjectures in [5] that the strategy for constructing $\lambda^{\mathcal{V}/\mathcal{V}}$ can be adapted to construct a weak distributive law $\mathcal{RV} \Rightarrow \mathcal{VR}$, where $\mathcal{R}$ is the Radon monad – a topological analogue of the distribution monad. But we show that this conjecture does not hold, the main problem being that the category of free algebras of the Vietoris monad is a strict subcategory of that of closed relations between compact Hausdorff spaces. To get a topological analogue of $\lambda^{\mathcal{D}/\mathcal{P}}$, we thus choose to study weak distributive laws between powerspace monads in the category of stably compact spaces instead [10], as in this category the closed relations match exactly the free algebras of a powerspace monad [8].

An important property of a stably compact space (X, τ) is that it comes with its de Groot dual: X can also be equipped with the topology τ^d whose open sets are the compact saturated subsets of (X, τ). A continuous map $f : X \to Y$ that is also continuous for the dual topologies is then said to be *proper*. There are then two notions of powerspaces, dual to one another: the Smyth powerspace of a stably compact space X is the space $\mathcal{Q}X$ of its compact saturated subsets equipped with the upper Vietoris topology, while its Hoare powerspace is the space $\mathcal{H}X$ of its closed subsets equipped with the lower Vietoris topology. It is well-known that $\mathcal{Q}X^d = (\mathcal{H}X)^d$: we also show that the two monads themselves are de Groot duals of one another, so that their unit and multiplicative laws are proper maps.

The main contribution of this work is to construct a weak distributive law $\mathcal{Q}\mathcal{Q} \Rightarrow \mathcal{Q}\mathcal{Q}$, using a known general construction starting from the identity monad morphism $\mathcal{Q} \Rightarrow \mathcal{Q}$, and a (strong) distributive law $\mathcal{Q}\mathcal{H} \Rightarrow \mathcal{H}\mathcal{Q}$, derived by hand, and which happens to be an isomorphism of monads $\mathcal{Q}\mathcal{H} \cong \mathcal{H}\mathcal{Q}$ with inverse the dual distributive law $\mathcal{H}\mathcal{Q} \Rightarrow \mathcal{Q}\mathcal{H}$, so that $(\mathcal{Q}\mathcal{H}X)^d = \mathcal{H}\mathcal{Q}X^d \cong \mathcal{Q}\mathcal{H}X^d$ and $\mathcal{Q}\mathcal{H}$ is self-dual.

A third kind of powerspace, the Plotkin powerspace, arises naturally as a combination of the Hoare and Smyth powerspaces: there is hope that making this combination categorical would also allow for combining the two weak laws above into a weak law for the Plotkin powerspace monad. Another next step would also be to extend the combination of probabilistic powerspaces over stably compact spaces with the Hoare and Smyth powerspaces [3, 4] to the monadic setting, again by constructing weak distributive laws.

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Craig Interpolation from Horn Semantics

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In this work we consider semantics of a logic in a class of first order structures axiomatized by universal Horn sentences, a *Horn class*. We give conditions on such a semantics which ensure that an amalgamation property for the Horn class implies Craig interpolation for the logic.

This generalizes the well-known result that for an algebraizable logic the amalgamation property for the associated class of algebras implies Craig interpolation.

κ-Horn classes and lattices of atomic Horn formulas

Let κ be a regular cardinal. Let Σ be a signature consisting of function symbols and Σ^+ an expansion of Σ by relation symbols.

- A *universal strict basic κ-Horn sentence* is a sentence of the form $\forall \vec{x}\colon \bigwedge_{i \in I} P_i(\vec{x}) \to P(\vec{x})$, with P_i, P atomic formulas over Σ^+ not equivalent to $\bot$ and $|I| < \kappa$.
- A κ-Horn theory is a theory axiomatized by universal strict basic κ-Horn sentences.
- A κ-Horn class is a class of Σ^+-structures axiomatized by a κ-Horn theory.

We shall say that a class $\mathbf{K}$ of Σ^+-structures has the *atomic amalgamation property* if given $A, B, C \in \mathbf{K}$ and maps $i_B\colon A \to B$, $i_C\colon A \to C$ that preserve and reflect the validity of atomic formulas (atomic embeddings), there exist a Σ^+-structure $D \in \mathbf{K}$ and atomic embeddings $e_B : B \to D$, $e_C : C \to D$ such that $e_B \circ i_B = e_C \circ i_C$.

For a κ-Horn theory $\mathbb{T}$, we define a lattice of atomic Horn formulas that will replace the congruence lattice from algebraic semantics: For a Σ-structure A let

- $G^{\mathbb{T}}(A) := \{(\theta, S) \mid \theta$ is a Σ-congruence on A and S an interpretation of $\mathfrak{R}$ on A/θ s.t. the resulting Σ^+-structure on A/θ is a $\mathbb{T}$-model$\}$
- We define an order on $G^{\mathbb{T}}(A)$ by declaring $(\theta, S) \leq (\theta', S')$ iff $\theta \subseteq \theta'$ and the induced quotient map $q_{\theta\theta'} : A/\theta \twoheadrightarrow A/\theta'$ is a homomorphism of Σ^+-structures for the interpretations S, S'.

$G^{\mathbb{T}}(A)$ is a κ-algebraic lattice.

κ-Horn Semantics

Let L be a logic over a signature Σ. Recall that an algebraic semantics for a logic L is a translation from formulas of L to sets of equations over the signature of L (i.e. atomic formulas of the first order language associated to Σ), commuting with substitution, and such that inference in the logic under this translation corresponds exactly to inference in the equational logic in a quasivariety $\mathbf{K}$.

This situation has been abstracted into the notion of *filter pair* in [AMP1], [AMP2], [AMP3]: A filter pair is a functor $G\colon \Sigma\text{-Str} \to \kappa\text{-AlgLat}$ together with a natural transformation to the power set functor $i\colon G \to \wp$, which objectwise preserves infima and κ-directed suprema. In the case of algebraic semantics for the functor one takes $G := Co_{\mathbf{K}}(\mathrm{Fm}) := \{\theta \mid \mathrm{Fm}/\theta \in \mathbf{K}\}$.

Horn Semantics arises by replacing the congruence lattice with the above lattice of atomic formulas of an expansion Σ^+ of Σ.

Theorem Let τ be a set of *atomic* Σ^+-formulas with at most one free variable, such that $|\tau| < \kappa$. The collection of maps $i^\tau = (i^\tau_A)_{A\in\Sigma-Str}$, defined by

$$\begin{array}{rcl} i^\tau_A : G^{\mathbb{T}}(A) & \to & (\mathcal{P}(A), \subseteq) \\ (\theta, S) & \mapsto & \{a \in A \mid \forall\varphi(x) \in \tau\colon\ A/(\theta, S) \vDash \varphi(a)\} \end{array}$$

is a natural transformation and for any $A \in \Sigma-Str$, i^τ_A preserves arbitrary infima and κ-directed suprema. In other words, $(G^{\mathbb{T}}, i^\tau)$ is a κ-filter pair.

Such a filter pair is called *Horn filter pair.*

Definition A *κ-Horn Semantics* for a logic L is a Horn filter pair whose image over the formula algebra is the lattice of theories of L. It is an *equivalent Horn Semantics* if the natural transformations are injective.

Examples

- For $\Sigma^+ = \Sigma$ a Horn semantics is precisely an algebraic semantics, and an equivalent Horn semantics corresponds precisely to an algebraizable logic.
- For $\Sigma^+ = \Sigma \cup \{F\}$ an expansion of the signature with a unary relation symbol one can define an equivalent Horn semantics corresponding to matrix semantics.
- For $\Sigma^+ = \Sigma \cup \{\leq\}$ an expansion of the signature with an inequality symbol, and a Horn theory demanding that this be an order relation, a Horn semantics is precisely an order algebraic semantics, and an equivalent Horn semantics corresponds precisely to an order algebraizable logic in the sense of [**Raf**]

Using the formalism of filter pairs, we can prove a general Craig Interpolation result:

Theorem Let $(G^{\mathbb{T}}, i^\tau)$ be a Horn semantics for a logic L. Suppose that the filter pair $(G^{\mathbb{T}}, i^\tau)$ has the "theory lifting property". If $\mathbf{K} := \mathrm{Mod}(\mathbb{T})$ has the atomic amalgamation property, then the logic L associated to $(G^{\mathbb{T}}, i^\tau)$ has the Craig entailment property.

The "theory lifting property" is a technical condition, satisfied by every filter pair presenting an equivalent Horn Semantics, but also in other cases.

Examples The above theorem specializes to the following statements:

- the well-known statement that for algebraizable logics, the amalgamation property entails Craig interpolation
- the well-known statement that the theory amalgamation property entails Craig interpolation
- a corresponding statement for order algebraizable logics
- the statement that for logics with an algebraic semantics in a regular variety, the amalgamation property entails Craig interpolation

In the talk we will review the notion of filter pair and explain the above results and examples.

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Remarks on the DeMorganization of a locale

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Regular subobjects (equivalently, extremal subobjects) in the category of locales are known as sublocales, and therefore they are the point-free counterparts of classical subspaces of a space. In a topological space X, every subspace induces a sublocale of its frame of opens $\Omega(X)$, but this correspondence is, in general, not one-to-one nor onto (see [7] for more information on the relation between sublocales and subspaces).

It is well known that every locale has a largest (in fact, unique) Boolean dense sublocale, which coincides with the least dense sublocale [4] — the Booleanization of the locale [1]. This is typically a pointless locale, in the sense that for any Hausdorff space without isolated points it does not contain any points at all.

Moreover, Caramello [2] (cf. also [3]) showed that every topos has a largest dense De Morgan subtopos. By applying it to toposes of sheves over locales, this immediately implies that every locale has a largest dense De Morgan sublocale, where we recall that a locale is said to be *De Morgan* or *extremally disconnected* if the identity

$$(a \wedge b)^* = a^* \vee b^*$$

holds for all $a, b \in L$ (see [5] for more information and other equivalent conditions).

The study of the DeMorganization directly for locales recently started in [6], where a direct proof of its existence was given using nuclei.

In this talk, we will give a direct, simpler, proof of the existence of the DeMorganization in terms of sublocales as concrete subsets, represented as in [7]. This helps understand the topological nature of the DeMorganization. Among others, we will show that, similarly to the Booleanization, the DeMorganization is also a fitted sublocale — i.e. one which occurs as an intersection of open sublocales.

Using these techniques, we will show that for every metric space without isolated points its DeMorganization coincides with its Booleanization (a proof was announced in the abstract of [6], but it was not materialized during the talk).

Time permitting, we will also look at analogues for infinite variants of De Morgan law in locale theory.

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On Non-Archimedean frames

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Frames (locales, complete Heyting algebras) are complete lattices L such that the following distributivity holds:

$$a \wedge \left(\bigvee X\right) = \bigvee\{a \wedge x \mid x \in X\}$$

for each $a \in L$ and $X \subseteq L$.

Frames can be understood as an algebraic manifestation of a topological space. Indeed, for every topological space S, the lattice of open sets $\mathcal{O}S$ constitute a frame but not every frame is a topology. A decent analysis of the category of topological spaces can be done in the language of frames (see [PP12]). An example of the above statement is the core of this talk:

Definition. A *non-archimedean* topological space S is a Hausdorff space with a base $\mathcal{B}$ satisfying the *trichotomy* laws: If $B_1, B_2 \in \mathcal{B}$, we have that either $B_1 \cap B_2 = \varnothing$ or $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$ holds (see, e.g., [Nyi75], [NR75], [Nyi99]).

Motivated by this, we introduce:

Definition. A frame is *non-archimedean* if it has a (non-archimedan) base $\mathcal{B}$ that satisfies these trichotomy laws: If $b_1, b_2 \in \mathcal{B}$, then either $b_1 \wedge b_2 = 0$ or $b_1 \leq b_2$ or $b_2 \leq b_1$ holds.

One of the main examples of non-archimedian frames comes from non-archimedian fields, by example consider the frame of p-adic numbers $\mathcal{L}(\mathbb{Q}_p)$, defined by [Ávi20], where $\mathbb{Q}_p$ is the field of p-adic numbers. This field was determined by Kurt Hensel in 1904 in analogy with the Laurent series $\mathbb{C}((t))$. Furthermore, this field is a non-archimedean field with the p-adic norm $|\cdot|_p$ defined over it. Since the set of open balls centered at rationals generates the open subsets of $\mathbb{Q}_p$, we consider these balls' (lattice) properties and think of them as generators. Thus, we can define the $\mathbb{Q}_p$ frame as follows.

Let $\mathcal{L}(\mathbb{Q}_p)$ be the frame generated by the elements $B_r(a)$, where $a \in \mathbb{Q}$ and $r \in |\mathbb{Q}| := \{p^{-n}, n \in \mathbb{Z}\}$, subject to the following relations:

(1) $B_r(a) \wedge B_s(b) = 0$ whenever $|a-b|_p \geq r \vee s$.

(2) $1 = \bigvee\{B_r(a) : a \in \mathbb{Q}, r \in |\mathbb{Q}|\}$.

(3) $B_r(a) = \bigvee\{B_s(b) \colon |a-b|_p < r, s < r, r \in |\mathbb{Q}|\}$.

Note that relation (3) implies that the set

$$\mathcal{B} := \{B_r(a) : r \in| \mathbb{Q}|, a \in \mathbb{Q}\}$$

is a base for $\mathcal{L}(\mathbb{Q}_p)$. Moreover, let $B_r(a)$, $B_s(b)$ be any two elements in $\mathcal{L}(\mathbb{Q}_p)$ and, without loss of generality, assume that $s \leq r$. Then, If $|a-b|_p \geq r$, we have $B_r(a) \wedge B_s(b) = 0$ by relation (1), and if $|a-b|_p < r$, we have $B_s(b) \leq B_r(a)$ by relation (3). Thus, for any $B_r(a), B_s(b) \in \mathcal{B}$, either

$$B_r(a) \wedge B_s(b) = 0 \text{ or } B_s(b) \leq B_r(a) \text{ or } B_s(b) \geq B_r(a).$$

It follows that $\mathcal{B} := \{B_r(a) : r \in |\mathbb{Q}|, a \in \mathbb{Q}\}$ is a non-archimedean base for the frame $\mathcal{L}(\mathbb{Q}_p)$, and that $\mathcal{L}(\mathbb{Q}_p)$ is a non-archimedean frame.

Another example of a similar nature of a non-archimedean frame is the frame of the Cantor set, denoted by $\mathcal{L}(\mathbb{Z}_p)$ [ÁUZ22].

As the examples show, the bases of these non-archimedean frames constitute a tree. This phenomenon is not a coincidence; in [Nyi99, Theorem 2.10] the author shows that every non-archimedean space is a subspace of a branch space of a tree. In the point-free context we have:

Theorem. Let A be a non-archimedean frame with base $\mathcal{B}$. Then A has a tree-base.

Since for every tree we have its branch space we have furthermore:

Theorem. A frame A is non-archimedean if and only if A is a quotient of a topology of a branch space of a tree.

In this talk we will give some details on these theorems and their connection with the spatiality of certain quotients of the Alexandroff topology given by the tree-base.

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The limits of prenexation in first-order Gödel logics

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One of the first recognised characteristics of classical logic is the existence of a prenex form for each formula. The quantifier-shifting rules are used non-uniquely to construct these prenex forms. The expressive power of prenex fragments is easy to see in classical logic because it coincides with the whole logic, and in Intuitionistic logic since the prenex formulas are very weak (the validity of the prenex formula is decidable). However, because Gödel logics are intermediary logics, the expressibility of its prenex is relatively important.

It is clear that prenex normal forms cannot be constructed in the usual sense in Gödel logics because some of the quantifier-shift rules may fail, but this does not imply that no prenex normal form exists. However, demonstrating that such prenex forms do not exist is more difficult. Prenexation does not work for $G_{[0,1]}$ when 0 is not isolated, since the formula $(\neg\forall x A(x) \wedge \forall x \neg\neg A(x))$ does not allow a prenex normal form. To prove this fact, we use a glueing argument. This result can be extended to all Gödel logics where there is one accumulation point from above, even if it is not 0.

In this talk we provide the complete classification for the first-order Gödel logics with respect to the property that the formulas admit logically equivalent prenex normal forms. We show that the only first-order Gödel logics that admit such prenex forms are those with finite truth value sets since they allow all quantifier-shift rules and the logic $G_\uparrow$ with only one accumulation point at 1. In all the other cases, there are, in general, no logically equivalent prenex normal forms. We will also see that $G_\uparrow$ is the intersection of all finite first-order Gödel logics.

The second stage of our research investigates the existence of the validity equivalent prenex normal form. Gödel logics with a finite truth value set admit such prenex forms. Gödel logics with an uncountable truth value set have the prenex normal form if and only if every surrounding of 0 is uncountable or 0 is an isolated point. Otherwise, uncountable Gödel logics are incomplete, and the prenex fragment is always complete with respect to the uncountable truth value set. Therefore, there is no effective translation to the valid formula and the valid prenex form. The countable case, however, is still up for debate.

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Monoidal aspects of cocomplete $\mathcal{V}$-enriched categories

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Fifty years ago, Lawvere's observation that ordered sets and metric spaces can be seen as enriched categories over a quantale opened a wide path for the quantitative theory of domains, using and generalising ideas from category theory, algebra, logic and and topology. Among the most beneficial and pleasing properties of such quantale-enriched categories are undoubtedly (co)completeness with respect to certain classes of (co)limits and commutation of such limits and colimits (with prominent examples featuring the ordered case, like continuous or completely distributive lattices).

The present talk considers the category $\mathcal{V}\text{-}\mathsf{Sup}$ of cocomplete enriched categories over a commutative quantale $\mathcal{V}$ – the $\mathcal{V}$-valued analogue of complete sup-lattices. The arrows are cocontinuous $\mathcal{V}$-functors. Cocomplete $\mathcal{V}$-categories are the Eilenberg-Moore algebras for the free cocompletion monad $\mathbb{D}$ on $\mathcal{V}$-categories. $\mathcal{V}\text{-}\mathsf{Sup}$ is *symmetric monoidal closed*. The corresponding tensor product $\otimes_{\mathcal{V}\text{-}\mathsf{Sup}}$ arises naturally, using that $\mathbb{D}$ is a KZ-monad (hence commutative) [6, 8], and it classifies bimorphisms [1, 7]. The internal hom is $\mathcal{V}\text{-}\mathsf{Sup}(A, B)$. Besides being monoidal closed, $\mathcal{V}\text{-}\mathsf{Sup}$ is also $*$-*autonomous*, with dualizing object $\mathcal{V}^{\mathsf{op}}$ [3]. Consequently, the tensor product $A \otimes_{\mathcal{V}\text{-}\mathsf{Sup}} B$ of two cocomplete $\mathcal{V}$-categories A and B can alternatively be characterised as $\mathcal{V}\text{-}\mathsf{Sup}(A, B^{\mathsf{op}})^{\mathsf{op}}$. In case of complete sup-lattices, this is precisely the set of Galois maps between them [10]. An alternative description of the tensor product of two complete sup-lattices is obtained as the set of all down-sets in their cartesian product that are join-closed in either coordinate [13]. Taking advantage of the 2-categorical setting, we can extend this description to (co)complete quantale-enriched categories.

Proposition 1. *Let A and B be two cocomplete $\mathcal{V}$-categories. Then the coreflexive inverter in $\mathcal{V}\text{-}\mathsf{Sup}$ of the 2-cell (inequality) $\mathbb{D}(\mathsf{y}_A \otimes \mathsf{y}_B) \Rightarrow \mathbb{D}_\forall(\mathsf{y}_A \otimes \mathsf{y}_B)$ is $A \otimes_{\mathcal{V}\text{-}\mathsf{Sup}} B$:*

$$\begin{tikzcd} A \otimes_{\mathcal{V}\text{-}\mathsf{Sup}} B \arrow[r] & \mathbb{D}(A \otimes B) \arrow[r, shift left, "\mathbb{D}(\mathsf{y}_A \otimes \mathsf{y}_B)"] \arrow[r, shift right, "\mathbb{D}_\forall(\mathsf{y}_A \otimes \mathsf{y}_B)"'] \arrow[r, phantom, "\Downarrow"] & \mathbb{D}(\mathbb{D}(A) \otimes \mathbb{D}(B)) \end{tikzcd}$$

In the above, $\mathsf{y}_A : A \to \mathbb{D}(A)$ and $\mathsf{y}_B : B \to \mathbb{D}(B)$ denote the Yoneda embeddings, and $\mathbb{D}_\forall(\mathsf{y}_A \otimes \mathsf{y}_B)$ is the right adjoint to the right adjoint to $\mathbb{D}(\mathsf{y}_A \otimes \mathsf{y}_B)$.

Nuclearity (in modern parlance *dualisability*) originally arose in Operator Theory, in order to mimic finite dimensionality behaviour (for objects) and matrix calculus (for arrows) [4]. It was subsequently observed that nuclearity was in fact a categorical concept, and that it could be defined in the more general context of (symmetric) monoidal closed categories: An arrow $f : A \to B$ is *nuclear* iff the associated $\mathbb{1} \to [A, B]$ factorises through $A^* \otimes B$, where $A^* = [A, \mathbb{1}]$, and an object A is *nuclear* if id_A is so [5]. Equivalently, $B \otimes A^* \cong [A, B]$ holds for all objects B. The nuclear objects in the category of sup-lattices and join-preserving maps are precisely the *completely distributive lattices* [5, 11, 12]. The analogue concept in the realm of $\mathcal{V}$-enriched categories, the cocomplete and completely distributive $\mathcal{V}$-categories ($\mathcal{V}\text{-}\mathsf{ccd}$) [14], are (co)complete $\mathcal{V}$-categories for which taking suprema distributes over limits, equivalently, they are the projective objects in $\mathcal{V}\text{-}\mathsf{Sup}$. Their behaviour with respect to the monoidal structure of $\mathcal{V}\text{-}\mathsf{Sup}$ is described in the next Proposition:

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Proposition 2. *1. The tensor product in $\mathcal{V}$-Sup of two $\mathcal{V}$-ccds is $\mathcal{V}$-ccd.*

2. The nuclear objects in $\mathcal{V}$-Sup are precisely the $\mathcal{V}$-ccds.

To each arbitrary category A canonically corresponds a cocomplete one, namely $\mathbb{D}A$. Being free in $\mathcal{V}$-Sup, the latter is also $\mathcal{V}$-ccd. To obtain more examples of (completely distributive) cocomplete $\mathcal{V}$-categories associated to A, consider the Isbell adjunction $[A^{\mathsf{op}}, \mathcal{V}] \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} [A, \mathcal{V}]^{\mathsf{op}}$. Taking the fixed points of this adjunction produces a $\mathcal{V}$-category $\mathbb{I}(A)$ into which A embeds, known as the *Isbell completion*, the categorical analogue of the MacNeille completion by cuts of a poset. As such, $\mathbb{I}(A)$ is a (complete and) cocomplete $\mathcal{V}$-category, hence an object of $\mathcal{V}$-Sup. When $\mathcal{V}$ is the two-element quantale, A is just an ordered set, and $\mathbb{I}(A)$ is a complete sup-lattice, which moreover is completely distributive if and only if the negation of the underlying order of A is a regular relation [2]. Regularity is a concept definable not only for relations; it can apply to arrows in an arbitrary category [9]. We seek to generalise this result to $\mathcal{V}$-categories. However, a quantale has only tensor product and internal hom. Properly handling negation in a quantale requires extra assumptions. We shall assume that $\mathcal{V}$ is a Girard quantale, the posetal analogue of a $*$-autonomous category. Then taking internal homs into the cyclic dualising element of $\mathcal{V}$ determines a negation operation $\neg$ on $\mathcal{V}$, in particular, on all $\mathcal{V}$-valued relations. Under these assumptions, we obtain the sought generalisation:

Theorem 3. *Let $\mathcal{V}$ be a Girard integral quantale and $\mathbb{I}(A)$ the Isbell completion of a $\mathcal{V}$-enriched category A. Then the following are equivalent:*

1. *$\mathbb{I}(A)$ is completely distributive (as a cocomplete $\mathcal{V}$-category), that, is, $\mathbb{I}(A)$ is a nuclear object in the $*$-autonomous category $\mathcal{V}$-Sup.*
2. *Negation $\neg A(-,-)$ of the $\mathcal{V}$-hom of $\mathcal{A}$ is regular as a $\mathcal{V}$-relation.*

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Intuitionistic modal logics: a minimal setting

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Abstract

We make a clean sweep of the tradition in intuitionistic modal logics by considering a new truth condition of $\Diamond$-formulas saying that in model $(W,\leq,R,V)$, $\Diamond A$ holds at $s{\in}W$ if there exists $t{\in}W$ where A holds and such that $s{\geq}{\circ}Rt$. While keeping the truth condition of $\Box$-formulas that is commonly used, we axiomatize validity in the class of all models. The resulting logic is the intuitionistic modal logic that we want to put forward as a candidate for the title of "minimal intuitionistic modal logic".

1 Syntax and semantics

Let **At** be a set of *atoms* (p, q, etc). The set **Fo** of all *formulas* (A, B, etc) is defined by $A ::= p|(A{\to}A)|\top|\bot|(A{\wedge}A)|(A{\vee}A)|\Box A|\Diamond A$. For all $A{\in}\mathbf{Fo}$, $\neg A$ is the abbreviation for $(A{\to}\bot)$.

A *Kripke frame* or a *KF* is a structure of the form $(W,\leq,R)$ where W is a nonempty set, $\leq$ is a partial order on W and R is a binary relation on W. Let $\mathcal{C}^{\mathbf{kf}}_{\mathbf{all}}$ be the class of all KFs. A KF $(W,\leq,R)$ is *forward (respectively: backward; downward) confluent* if for all $s,t{\in}W$, if $s{\geq}{\circ}Rt$ then $sR{\circ}{\geq}t$ (respectively: for all $s,t{\in}W$, if $sR{\circ}{\leq}t$ then $s{\leq}{\circ}Rt$; for all $s,t{\in}W$, if $s{\leq}{\circ}Rt$ then $sR{\circ}{\leq}t$). Let $\mathcal{C}^{\mathbf{kf}}_{\mathbf{fc}}$ (respectively: $\mathcal{C}^{\mathbf{kf}}_{\mathbf{bc}}$; $\mathcal{C}^{\mathbf{kf}}_{\mathbf{dc}}$; $\mathcal{C}^{\mathbf{kf}}_{\mathbf{fbc}}$; $\mathcal{C}^{\mathbf{kf}}_{\mathbf{fdc}}$; $\mathcal{C}^{\mathbf{kf}}_{\mathbf{bdc}}$; $\mathcal{C}^{\mathbf{kf}}_{\mathbf{fbdc}}$) be the class of all forward (respectively: backward; downward; forward and backward; forward and downward; backward and downward; forward, backward and downward) confluent KFs. A *valuation on a KF* $(W,\leq,R)$ is a function $V : \mathbf{At} \longrightarrow \wp(W)$ associating a $\leq$-closed subset of W to each atom. Such a function can be extended as a function $V : \mathbf{Fo} \longrightarrow \wp(W)$ associating to each $A{\in}\mathbf{Fo}$ a $\leq$-closed subset $V(A)$ of W defined as usual when either A is an atom, or the main connective of A is intuitionistic and as follows otherwise: (**i**) $V(\Box A)=\{s{\in}W$: for all $t{\in}W$, if $s{\leq}{\circ}Rt$ then $t{\in}V(A)\}$; (**ii**) $V(\Diamond A)=\{s{\in}W$: there exists $t{\in}W$ such that $s{\geq}{\circ}Rt$ and $t{\in}V(A)\}$. A *relational model* is a couple consisting of a KF and a valuation on that KF. *Truth in a relational model, validity in a KF* and *validity on a class of KFs* are defined as usual. For all classes $\mathcal{C}$ of KFs, let $\mathtt{Log}(\mathcal{C})$ be the *logic of* $\mathcal{C}$.

A *H-modal algebra* or a *HMA* is a structure of the form $(H,\leq_H,\to_H,\Box_H,\Diamond_H)$ where $(H,\leq_H,\to_H)$ is a Heyting algebra and $\Box_H : H{\longrightarrow}H$ and $\Diamond_H : H{\longrightarrow}H$ are operators such that for all $a,b,c{\in}H$: (**i**) $\Box_H\top_H=\top_H$; (**ii**) $\Box_H(a{\wedge_H}b)=\Box_H a{\wedge_H}\Box_H b$; (**iii**) $\Diamond_H\bot_H=\bot_H$; (**iv**) $\Diamond_H(a{\vee_H}b)=\Diamond_H a{\vee_H}\Diamond_H b$; (**v**) if $\Diamond_H a{\leq_H}b{\vee_H}\Box_H(a{\to_H}c)$ then $\Diamond_H a{\leq_H}b{\vee_H}\Diamond_H c$. Let $\mathcal{C}^{\mathbf{hma}}_{\mathbf{all}}$ be the class of all HMAs. A HMA $(H,\leq_H,\to_H,\Box_H,\Diamond_H)$ is *forward (respectively: backward; downward) confluent* if for all $a,b{\in}H$, $\Diamond_H(a{\to_H}b){\leq_H}(\Box_H a{\to_H}\Diamond_H b)$ (respectively: $(\Diamond_H a{\to_H}\Box_H b){\leq_H}\Box_H(a{\to_H}b)$; $\Box_H(a{\vee_H}b){\leq_H}\Diamond_H a{\vee_H}\Box_H b$). Let $\mathcal{C}^{\mathbf{hma}}_{\mathbf{fc}}$ (respectively: $\mathcal{C}^{\mathbf{hma}}_{\mathbf{bc}}$;

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$\mathcal{C}^{\mathbf{hma}}_{\mathbf{dc}}$; $\mathcal{C}^{\mathbf{hma}}_{\mathbf{fbc}}$; $\mathcal{C}^{\mathbf{hma}}_{\mathbf{fdc}}$; $\mathcal{C}^{\mathbf{hma}}_{\mathbf{bdc}}$; $\mathcal{C}^{\mathbf{hma}}_{\mathbf{fbdc}}$) be the class of all forward (respectively: backward; downward; forward and backward; forward and downward; backward and downward; forward, backward and downward) confluent HMA. A *valuation on a HMA* $(H, \leq_H, \rightarrow_H, \Box_H, \Diamond_H)$ is a function $V : \mathbf{At} \longrightarrow H$ associating an element of H to each atom. Such a function can be extended as a function $V : \mathbf{Fo} \longrightarrow H$ associating to each $A \in \mathbf{Fo}$ an element $V(A)$ of H defined as usual when either A is an atom, or the main connective of A is intuitionistic and as follows otherwise: (**i**) $V(\Box A) = \Box_H V(A)$; (**ii**) $V(\Diamond A) = \Diamond_H V(A)$. An *algebraic model* is a couple consisting of a HMA and a valuation on that HMA. *Truth in an algebraic model, validity in a HMA* and *validity on a class of HMAs* are defined as usual. For all classes $\mathcal{C}$ of HMAs, let $\mathtt{Log}(\mathcal{C})$ be the *logic of* $\mathcal{C}$.

2 Axiomatization and completeness

An *intuitionistic modal logic* is a set of formulas closed for uniform substitution, containing the standard axioms of **IPL**, closed with respect to the standard inference rules of **IPL**, containing the axioms $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$, $\Box(p \vee q) \rightarrow ((\Diamond p \rightarrow \Box q) \rightarrow \Box q)$, $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$ and $\neg\Diamond\bot$ and closed with respect to the inference rules $\frac{p}{\Box p}$, $\frac{p \leftrightarrow q}{\Diamond p \leftrightarrow \Diamond q}$ and $\frac{\Diamond p \rightarrow q \vee \Box(p \rightarrow r)}{\Diamond p \rightarrow q \vee \Diamond r}$. We also consider the axioms (**Af**) $\Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q)$, (**Ab**) $(\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$ and (**Ad**) $\Box(p \vee q) \rightarrow \Diamond p \vee \Box q$. Let $\mathbf{L}_{\min}$ be the least intuitionistic modal logic. For all intuitionistic modal logics $\mathbf{L}$ and for all $A \in \mathbf{Fo}$, let $\mathbf{L} \oplus A$ be the least intuitionistic modal logic containing $\mathbf{L}$ and A. Let $\mathbf{L}_{\mathbf{fc}}$ (respectively: $\mathbf{L}_{\mathbf{bc}}$; $\mathbf{L}_{\mathbf{dc}}$; $\mathbf{L}_{\mathbf{fbc}}$; $\mathbf{L}_{\mathbf{fdc}}$; $\mathbf{L}_{\mathbf{bdc}}$; $\mathbf{L}_{\mathbf{fbdc}}$) be $\mathbf{L}_{\min} \oplus (\mathbf{Af})$ (respectively: $\mathbf{L}_{\min} \oplus (\mathbf{Ab})$; $\mathbf{L}_{\min} \oplus (\mathbf{Ad})$; $\mathbf{L}_{\min} \oplus (\mathbf{Af}) \oplus (\mathbf{Ab})$; $\mathbf{L}_{\min} \oplus (\mathbf{Af}) \oplus (\mathbf{Ad})$; $\mathbf{L}_{\min} \oplus (\mathbf{Ab}) \oplus (\mathbf{Ad})$; $\mathbf{L}_{\min} \oplus (\mathbf{Af}) \oplus (\mathbf{Ab}) \oplus (\mathbf{Ad})$).

Proposition 1.

- $\mathbf{L}_{\min} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{all}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{all}})$;
- $\mathbf{L}_{\mathbf{fc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{fc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{fc}})$; $\mathbf{L}_{\mathbf{bc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{bc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{bc}})$; $\mathbf{L}_{\mathbf{dc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{dc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{dc}})$;
- $\mathbf{L}_{\mathbf{fbc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{fbc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{fbc}})$; $\mathbf{L}_{\mathbf{fdc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{fdc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{fdc}})$; $\mathbf{L}_{\mathbf{bdc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{bdc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{bdc}})$;
- $\mathbf{L}_{\mathbf{fbdc}} = \mathtt{Log}(\mathcal{C}^{\mathbf{kf}}_{\mathbf{fbdc}}) = \mathtt{Log}(\mathcal{C}^{\mathbf{hma}}_{\mathbf{fbdc}})$.

Proposition 2.

- **WK** *[3] and* $\mathbf{L}_{\min}$ *are not comparable;*
- **WK** *[3] is strictly contained in* $\mathbf{L}_{\mathbf{fc}}$*;*
- $\mathbf{L}_{\mathbf{fc}}$ *and* **FIK** *[1] are equal;*
- $\mathbf{L}_{\mathbf{fbc}}$ *and* **IK** *[2] are equal;*
- $\mathbf{L}_{\mathbf{fbdc}}$ *is strictly contained in* **K** *— the least normal modal logic.*

All in all, $\mathbf{L}_{\min}$ is the intuitionistic modal logic that we want to put forward as a candidate for the title of "minimal intuitionistic modal logic".

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Characterizing finite measurable Boolean algebras*

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In our daily life, we are used to compare things. We sort physical objects from smaller to bigger, or propositions from less likely to more likely. These relations contribute to our intuitive understanding of reality, and are naturally represented by a pre-order on a Boolean algebra. Such framework is typically *qualitative*, as two elements can only be related in three possible manners: smaller, bigger, or equivalent – without any consideration of degree or magnitude. By contrast, human beings also managed to quantify some of their intuitions through *measurement*, with examples including length, volume, temperature, and probabilities. Quantitative reasoning is a core component of scientific inquiry, and its mathematical foundations have been studied extensively in [KLTS71]. Formally, a *measure* on a **finite**[1] Boolean algebra $B = (2^X, \cap, \cup, \cdot^c, 0, 1)$ is a map $\mu : B \to [0, \infty]$ satisfying $\mu(0) = 0$ and $\mu(a \cup b) = \mu(a) + \mu(b)$ whenever $a \cap b = 0$. We call μ *bounded* if in addition we have $\mu(a) < \infty$ for all $a \in B$. Obviously, a measure μ always induces a binary relation $\preceq_\mu$ on B, defined by $a \preceq_\mu b \iff \mu(a) \leq \mu(b)$. Relations of the form $\preceq_\mu$ will be called *measurable*, and *bounded measurable* in case μ is a bounded measure. So there is a direct bridge from quantitative to qualitative comparison, but the other way around is more limited, and this raises the question of which conditions on a binary relation $\preceq$ are necessary and sufficient for $\preceq$ to be (bounded) measurable. In the case of bounded measures, this problem was solved by Kraft, Pratt and Seidenberg in their 1959 paper [KPS59], and later rewritten by Scott [Sco64] in a clearer manner. We present their conditions below. Given $x \in X$ and $a_1, \ldots, a_m \in B$, we write $\mathtt{count}_x(a_1, \ldots, a_m) := \{i \in [1, m] : x \in a_i\}$.

Theorem 1. *A binary relation $\preceq$ on B is bounded measurable if and only if the following conditions are satisfied, for all $m \geq 1$ and for all $a, b, a_1, \ldots, a_m, b_1, \ldots, b_m \in B$:*

- **Positivity:** *$0 \preceq a$;*
- **Comparability:** *$a \preceq b$ or $b \preceq a$;*
- **Cancellation:** *if $\mathtt{count}_x(a_1, \ldots, a_m) = \mathtt{count}_x(b_1, \ldots, b_m)$ for all $x \in X$ and $a_i \preceq b_i$ for all $i \in [1, m-1]$, then $b_m \preceq a_m$.*

However, this result is not fully satisfying for a number a reasons, related to the cancellation conditions. First, they involve the high-level operator $\mathtt{count}_x$, and even though they can be rewritten in a purely Boolean manner [Seg71], they remain quite awkward to read and compute. Second, they come in infinite number, and thus fail to provide a finite axiomatization for various logics of measure, see for instance [Seg71, Gär75, vdH96]. It is surprising, perhaps, that this result has never been improved in sixty years, nor proved to be optimal. In this work, we break this uncomfortable status quo by proposing the following new characterization.

Theorem 2. *A binary relation $\preceq$ on B is bounded measurable if and only if the following conditions are satisfied, for all $a, b, c, d \in B$:*

*The preprint on which this talk is based can be found here: https://hal.science/hal-04544145.
†Speaker.
[1]The case of infinite Boolean algebras is much more complicated, so we only consider the finite case.

- **Comparability:** *$a \preceq b$ or $b \preceq a$;*
- **Linearity:** *if $a \cap c = 0$ and $a \cup c \preceq b \cup d$ and $d \preceq c$, then $a \preceq b$.*

Let us briefly sketch the proof of Theorem 2. The strategy for the right-to-left implication is to derive the conditions of Theorem 1 from comparability and linearity. Positivity follows from linearity with $a = 0$ and $b = c = d$. For cancellation, assume that $\texttt{count}_x(a_1, \dots, a_m) = \texttt{count}_x(b_1, \dots, b_m)$ for all $x \in X$, and that $a_i \preceq b_i$ for all $i \in [1, m-1]$. Consider for a moment the case where $a_1, \dots, a_m, b_1, \dots, b_m$ are all pairwise disjoint. Then, the counting assumption yields $b_1 \cup \dots \cup b_m \preceq a_1 \cup \dots \cup a_m$, and by applying linearity $m-1$ times we arrive at $b_m \preceq a_m$. This does not work in the general case, because when $\texttt{count}_x(a_1, \dots, a_m) \geq 2$, the large union $a_1 \cup \dots \cup a_m$ fails to keep track of the different repetitions of x. We can nonetheless bypass this issue, and fall back to the previous case, by 'duplicating' the elements of X. In a critical lemma, we show that we can introduce equivalent copies $x^1, \dots, x^{2m}$ of every $x \in X$, in a way that preserves positivity, comparability, and a weaker version of linearity. We then tweak the sets $a_1, \dots, a_m, b_1, \dots, b_m$ by replacing their members with corresponding copies, so that one copy never occurs twice (see the example below).

$$
\begin{array}{ll|ll}
a_1 = \{x, z\} & b_1 = \{x, y\} & a_1^* = \{x^1, z^1\} & b_1^* = \{x^5, y^5\} \\
a_2 = \{x, y\} & b_2 = \{z\} & a_2^* = \{x^2, y^1\} & b_2^* = \{z^5\} \\
a_3 = \{z\} & b_3 = \{x, y, z\} & a_3^* = \{z^2\} & b_3^* = \{x^6, y^6, z^6\} \\
a_4 = \{x\} & b_4 = \{x\} & a_4^* = \{x^3\} & b_4^* = \{x^7\}
\end{array}
$$

It then suffices to apply the previous reasoning to the sets $a_1^*, \dots, a_m^*, b_1^*, \dots, b_m^*$.

We also address the case of arbitrary measurable relations.

Theorem 3. *A binary relation $\preceq$ on B is measurable if and only if the following conditions are satisfied, for all $a, b, c, d \in B$:*

- **Comparability:** *$a \preceq b$ or $b \preceq a$;*
- **Transitivity:** *$a \preceq b$ and $b \preceq c$ implies $a \preceq c$;*
- **Monotonicity:** *$a \subseteq b$ implies $a \preceq b$;*
- **Bounded Linearity:** *if $1 \not\preceq c$ and $a \cap c = 0$ and $a \cup c \preceq b \cup d$ and $d \preceq c$, then $a \preceq b$.*

Finally, we observe that the conditions of Theorem 2 and Theorem 3 can be checked in space logarithmic in the size of B. In the case of bounded measurable relations, this is a direct improvement on the polynomial space algorithm of Kraft, Pratt and Seidenberg [KPS59].

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Category-theoretic Fraïssé theory: an overview

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Abstract

In the talk I will give an overview of countable abstract Fraïssé theory formulated in the language of category theory. We start with a core setup (free completion, characterization of the Fraïssé limit, existence of a Fraïssé sequence), demonstrate it on several examples, and then we sketch further directions in which the core setup can be extended.

Recall that a first-order structure U is *ultrahomogeneous* if every isomorphism $f\colon A \to B$ between finitely generated substructures $A, B \subseteq U$ can be extended to an automorphism $\tilde{f}\colon U \to U$. Most classical countable ultrahomogeneous structures include the *linear order of rationals*, the *random graph*, and the *rational Urysohn metric space*. Study of countable ultrahomogeneous structures goes back to Fraïssé [5] and so is sometimes called *Fraïssé theory*. Model-theoretic treatment is now classical, see e.g. [6].

In 2006, Irwin and Solecki [7] introduced *projective Fraïssé theory*, where instead of embeddings of first-order structures, quotients of topological structures are considered. The (projectively) homogeneous structure is obtained as a limit of an inverse sequence of quotient maps, instead of taking the union of an increasing chain. The particular limit obtained by Irwin and Solecki was the Cantor space endowed with a special closed equivalence relation with the quotient space being the *pseudo-arc*, a well-known continuum. Since then, many continua were realized as quotients of projective Fraïssé limits, see e.g. [11], [2].

It is natural to formulate Fraïssé theory using the language of category theory. This allows for clear and general proofs capturing the essence of the constructions involved. Extra structure like the induced topology of the automorphism group of the Fraïssé limit also arises naturally. Such treatment of Fraïssé theory provides a unified framework: there is essentially no difference between classical Fraïssé theory of first-order structures and projective Fraïssé theory of topological structures. It also provides flexibility: we can easily consider other morphisms than embeddings like left-invertible embeddings, embedding-projection pairs, relational morphisms, or abstract elements of a monoid.

Category-theoretical Fraïssé theory was pioneered by Droste and Göbel [4] who started with a *semi-algebroidal* category of "large" objects $\mathcal{L}$, proved the uniqueness of an $\mathcal{L}$-object homogeneous over the full subcategory $\mathcal{L}_{\text{fin}}$ of *finite objects*, and characterized its existence in the case when $\mathcal{L}_{\text{fin}}$ is essentially countable. On the other hand, Kubiś [9] started with a category of "small" objects $\mathcal{K}$ and introduced the notion of *Fraïssé sequence* in $\mathcal{K}$ (also of uncountable length), which serves as the Fraïssé limit in the category of sequences $\sigma\mathcal{K}$. In applications, $\sigma\mathcal{K}$ is identified with a particular category of large structures we are interested in. Existence of a Fraïssé sequence is closely connected to the notion of *dominating subcategory*, also introduced in [9]. The two views can be combined by working with a pair of categories $\mathcal{K} \subseteq \mathcal{L}$, as done by Caramello [3].

In the talk I shall give an overview of a polished framework. An $\mathcal{L}$-object U is *homogeneous* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every pair of $\mathcal{L}$-maps from a $\mathcal{K}$-object $f, g\colon x \to U$ there is an automorphism

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$h: U \to U$ such that $h \circ g = f$. We say that $\langle \mathcal{K}, \mathcal{L} \rangle$ is a *free completion* (or more precisely, free sequential cocompletion) if $\mathcal{L}$ essentially arises from $\mathcal{K}$ by freely adding colimits of $\mathcal{K}$-sequences. The core of countable abstract Fraïssé theory can be summarized by the following two theorems.

Theorem (Characterization of the Fraïssé limit)**.** Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

(1) U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,

(2) U is cofinal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,

(3) U is the $\mathcal{L}$-colimit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is Fraïssé in $\mathcal{K}$. Such U is called the *Fraïssé limit* of $\mathcal{K}$ in $\mathcal{L}$.

Theorem (Existence of a Fraïssé sequence)**.** Let $\mathcal{K}$ be a category. Then $\mathcal{K}$ has a Fraïssé sequence if and only if $\mathcal{K}$ is a *Fraïssé category*, i.e. $\mathcal{K} \neq \emptyset$ and

(1) $\mathcal{K}$ is directed,

(2) $\mathcal{K}$ has the amalgamation property,

(3) $\mathcal{K}$ has a countable dominating subcategory.

After explaining the core setup, we demonstrate it on several examples and see how it encompasses classical and projective situations, namely, how $\langle \mathcal{K}, \mathcal{L} \rangle$ being a free completion is verified in applications. Then we sketch several directions in which the core setup can be extended: *weak Fraïssé theory* [10], *metric-enriched categories* [8] and *MU-categories* [1], and *self-generic Fraïssé limits* in situations beyond free completion (joint work in progress with Matheus Duzi Ferreira Costa).

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The quasivariety $\mathbf{SP}(L_6)$. II. A duality result.

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We prove that the category of complete bialgebraic $(0,1)$-lattices belonging to the quasivariety $\mathbf{SP}(L_6)$ generated by a finite lattice L_6 with complete $(0,1)$-lattice homomorphisms, is dually equivalent to the category of so-called L_6-spaces with L_6-morphisms. It was established in [1] that the quasivariety $\mathbf{SP}(L_6)$ forms a variety and a finite equational basis for this variety was found. Our proof is based on the approach proposed by V. Dziobiak in [2,3].

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Enriched and Homotopical Coalgebra

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Coalgebra has emerged from the desire to find an abstraction of the behaviour of computational models [Rut00]. It starts with the insight that behaviour of many systems arises by repeated observation of a morphism $c\colon X \to FX$, where the kind of observations that can be made are determined by a functor $F\colon \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$. The idea is that FX is the space of possible observations on X that the coalgebra c yields. Instances of this view are transition systems, concurrent systems, probabilistic and timed systems, coinductive proofs, and various systems with topological structure, such as topological models of modal logic, dynamical systems and hybrid systems. A coalgebra c gives rise to behaviour in form of a sequence $X \xrightarrow{c} FX \xrightarrow{Fc} F(FX) \xrightarrow{F(Fc)} \cdots$ that recursively expands the observations. If this sequence approaches a limit, then this limit can be interpreted as total view on the behaviour of c [Bar93].

In this talk, I wish to present developments of enriched coalgebra in two main directions. The first direction is a theory of enriched categories and fibrations of coalgebras. Enriched category theory allows us to apply coalgebra to a wide variety of areas, which are not captured by categories with sets of morphisms. For instance, we can instead consider coalgebras in metric spaces, in order-enriched categories [BKPV11, BK11], topological or simplicial categories etc. In this direction, I aim to first present a few basic results and examples on enrichment, weighted (co)limits and (co)tensors for coalgebras. Then we turn to coalgebraic modal logic [CKP$^+$11, Mos99], which allows us to make partial observations on the recursive sequence mentioned above. Over plain categories, various correspondence results between bisimilarity and logical equivalence have been obtained [Kli07, Pat03, Sch08], and they have been extend to coalgebras in enriched categories [BD13, Wil12, Wil13]. Recently, it was shown how results in coalgebraic modal logic can be extended to other predicates by modelling the target predicate as a fibration map $(F, \overline{F})$ on a fibration $p\colon \mathcal{E} \to \mathcal{B}$, the modal logic as initial algebra for a functor L on a suitable category $\mathcal{D}$ of algebras, and the relation between the two by a pair of dual adjunction as in the diagram on the right [KR21]. Whenever the two adjunctions are related by distributive laws and $\mathcal{B}$ comes with a factorisation system, we can general obtain soundness and completeness results. My goal is to present an enriched version of this approach to enriched coalgebraic modal logic, where the fibration etc. are suitably enriched.

$$\begin{tikzcd}
\mathcal{E} \arrow[loop left, "\overline{F}"] \arrow[d, "p"] \arrow[drr, bend left, "\overline{P}"] & & \\
 & & \mathcal{D}^{\mathrm{op}} \arrow[ull, "\overline{Q}"] \arrow[dll, bend left, "Q"] \arrow[loop right, "L^{\mathrm{op}}"] \\
\mathcal{B} \arrow[loop left, "F"] \arrow[urr, "P"] & &
\end{tikzcd}$$

The second direction of development concerns enriched Kleisli categories. The Kleisli category of a monad is a well-known model for programs with computational effects. If the Kleisli category is enriched, then this enrichment provides an account of other computational features, such as recursion via CPO-enrichment. I will show how to obtain an $\mathcal{M}$-enrichment for the Kleisli category of a monad T on a category $\mathcal{V}$, even though $\mathcal{V}$ may not be $\mathcal{M}$-enriched, if the monad factor through the right-adjoint U of a suitable adjunction as in the diagram on the left. This result covers examples like order- and CPO-enrichment in case of the powerset and distribution monad that are typical in program semantics. We will also look at topological enrichment, which is the base of a homotopy theory for coalgebra, and can be used in topological models of modal logic [GT22, KKV04, Bal03, VdB22] and hybrid systems [NB18, Nev17].

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "T"] \arrow[dr, "G"'] & \mathcal{V} \arrow[d, bend right, "L"'] \arrow[d, phantom, "\dashv"] \\
 & \mathcal{M} \arrow[u, bend right, "U"']
\end{tikzcd}$$

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Non-distributive description logics *

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Basic non-distributive modal logic (a.k.a. *LE-logic*) is the non-distributive counterpart of positive modal logic without the distributivity axiom. Algebraically, it can be viewed as the logic of arbitrary lattices expanded with normal modal operators. Polarity-based semantics for LE-logic is given by a tuple $M = (\mathbb{F}, V)$, where $\mathbb{F} = (A, X, I, R_\Box, R_\Diamond)$ is an *enriched formal context* [3], i.e. , a *formal context* $\mathbb{P} = (A, X, I)$ enriched with I*-compatible* [3] relations $R_\Box \subseteq A \times X$ and $R_\Diamond \subseteq X \times A$, and V is a valuation map which maps LE-formulas to *formal concepts* defined by $\mathbb{P}$. Due to its natural connection with Formal Concept Analysis [4], LE-logic with its polarity-based semantics has been studied as the "logic of categorization" expanded with modal operators [3]. Motivated by this insight, in [5] we defined a two-sorted non-distributive *description logic* counterpart of LE-logic called LE-$\mathcal{ALC}$.

LE-$\mathcal{ALC}$ provides a natural description logic [1] to represent and reason about (partial) knowledge about formal contexts and concepts defined by them. LE-$\mathcal{ALC}$ has same concept names as LE-logic formulas, and has an analogous intended interpretation on the complex algebras of enriched formal contexts. This is similar to the classical case, where concept names of description logic are same as $\mathcal{ALC}$ and are interpreted over Kripke semantics in a similar manner.
Concept names in LE-$\mathcal{ALC}$ over a set of atomic concepts $\mathcal{D}$ are defined as follows:

$$C := D \in \mathcal{D} \mid C \wedge C \mid C \vee C \mid \top \mid \bot \mid \langle R_\Diamond \rangle C \mid [R_\Box]C,$$

As usual, $\vee$ and $\wedge$ are to be interpreted as the smallest common superconcept and the greatest common subconcept. The constants $\top$ and $\bot$ are to be interpreted as the largest and the smallest concept, respectively. Like in the classical case, modal operators can be assigned various interpretations such as knowledge or approximation [3, 2]. LE-$\mathcal{ALC}$ has individual names of two types OBJ and FEAT intended to be interpreted as object and features names, respectively. LE-$\mathcal{ALC}$ ABox assertions are of the form:

$$aR_\Box x, \quad xR_\Diamond a, \quad aIx, \quad a : C, \quad x :: C, \quad \neg\alpha,$$

where α is any of the first five ABox terms, and TBox assertions are of the form $C_1 \equiv C_2$ for two concept names C_1 and C_2. The intended interpretation of term $a : C$ (resp. $x :: C$) is object (resp. feature) a (resp. x) is an element of (resp. feature describing) C. Relational terms are interpreted in natural manner, and term $\neg\alpha$ is as negation of term α. Term $C_1 \equiv C_2$ is interpreted as concepts C_1 and C_2 are equivalent.

An *interpretation* for LE-$\mathcal{ALC}$ is a tuple $\mathrm{I} = (\mathbb{F}, \cdot^{\mathrm{I}})$, where $\mathbb{F} = (\mathbb{P}, \mathcal{R}_\Box, \mathcal{R}_\Diamond)$ is an enriched formal context, and $\cdot^{\mathrm{I}}$ maps:
1. individual names $a \in \mathsf{OBJ}$ (resp. $x \in \mathsf{FEAT}$), to some $a^{\mathrm{I}} \in A$ (resp. $x^{\mathrm{I}} \in X$);
2. relation names I, $R_\Box$ and $R_\Diamond$ to relations I^{I}, $R_\Box^{\mathrm{I}}$ and $R_\Diamond^{\mathrm{I}}$ in $\mathbb{F}$;
3. any primitive concept D to $D^{\mathrm{I}} \in \mathbb{F}^+$, and other concepts as follows:

$$\bot^{\mathrm{I}} = (X^{\downarrow}, X) \qquad \top^{\mathrm{I}} = (A, A^{\uparrow}) \qquad (C_1 \wedge C_2)^{\mathrm{I}} = C_1^{\mathrm{I}} \wedge C_2^{\mathrm{I}}$$
$$(C_1 \vee C_2)^{\mathrm{I}} = C_1^{\mathrm{I}} \vee C_2^{\mathrm{I}} \qquad ([R_\Box]C)^{\mathrm{I}} = [R_\Box^{\mathrm{I}}]C^{\mathrm{I}} \qquad (\langle R_\Diamond \rangle C)^{\mathrm{I}} = \langle R_\Diamond^{\mathrm{I}} \rangle C^{\mathrm{I}}$$

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An interpretation I is a *model* for an LE-$\mathcal{ALC}$ knowledge base $(\mathcal{A}, \mathcal{T})$ if $\mathrm{I} \models \mathcal{A}$ and $\mathrm{I} \models \mathcal{T}$. In [5], we proved the following theorem regarding the complexity of checking consistency of LE-$\mathcal{ALC}$ knowledge bases.

Theorem 1. *A tableaux algorithm exists for LE-$\mathcal{ALC}$, offering a sound and complete polynomial time decision procedure for verifying the consistency of LE-$\mathcal{ALC}$ knowledge bases by constructing a polynomial size model $Tab(\mathcal{K})$ for any consistent knowledge base.*

Several extensions of $\mathcal{ALC}$ with different concept constructors and axioms have been extensively researched. On our ongoing work, we generalized these results to extension of LE-$\mathcal{ALC}$ with axioms reflexivity, symmetery, and transitivity called LE-$\mathcal{ALCR}$ which can be seen as description logic for *rough concepts* [2]. We also proved similar results for extension of LE-$\mathcal{ALCR}$ with two new constructors: feature inconsistency pairs (i.e., pairs of features that no object can share) and concepts generated by sets of features.

Description logic ontologies play a crucial role in providing answers to queries based on incomplete databases. The following property of the model constructed by the Tableaux algorithm for LE-$\mathcal{ALC}$ is crucial with regards to querry answering over LE-$\mathcal{ALC}$.

Lemma 1. *Let $\mathcal{K} = (\mathcal{A}, \mathcal{T})$ be a consistent LE-$\mathcal{ALC}$ knowledge base with acyclic TBox. Let b, y, C, and C' be any concept names appearing in $\mathcal{T}$. Then for any term t consisting of individual, role, and concept names appearing in $\mathcal{K}$,*
$Tab(\mathcal{T}) \models t$ iff for every model I of $\mathcal{T}$, $\mathrm{I} \models t$.

Lemma 1 implies that many querries over LE-$\mathcal{ALC}$ knowledge bases like **ascription querries** ('does object b has feature y', 'name all the objects having feature y', etc.), **membership querries** ('does object b belong to concept C', 'name all the features defining concept C', etc.), **subsumption querries** ('Is concept C_1 included in C_2'?, 'Name all the concepts included in C_1') can be answered by only looking at the model $Tab(\mathcal{K})$. As $Tab(\mathcal{K})$ can be constructed in polynomial time and is of polynomial size (in size of $|\mathcal{K}|$), we can answer querries over LE-$\mathcal{ALC}$ knowledge bases with acyclic TBoxes in polynomial time.

We believe that similar approach can be used to answer more complex querries like *ontology equivalence querries* ('Are two given ontologies equivalent?') and to perform tasks like *querry-based ontology learning* in polynomial-time. We believe these results show that LE-$\mathcal{ALC}$ and its extensions allows us to solve many important reasoning tasks relating to knowledge representation and reasoning in relation to formal contexts and concepts efficiently.

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Induced congruences in σ-frames

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Just as frames generalise topological spaces ([3]), σ-frames generalise σ-topological spaces and, consequently, measurable spaces. Let us recall that a *σ-frame* [1] is a join-σ-complete lattice (that is, a lattice with countable joins) satisfying the distributive law

$$\Big(\bigvee_{a\in A} a\Big) \wedge b = \bigvee_{a\in A} (a \wedge b)$$

for every countable $A \subseteq L$ and $b \in L$. A map between σ-frames is called a *σ-frame homomorphism* if it preserves finite meets (including the top element 1 given by empty meet) and countable joins (including the bottom element 0 given by empty join).

Following Simpson [4], where a new approach to the problem of measuring subsets was proposed, we have been interested in approaching measure theory in the category of σ-frames and σ-frame homomorphisms. In a follow-up to our study of measurable functions in [2], we intend to investigate whether the notion of σ-sublocale generalises the notion of σ-subspace in a way similar to the case of sublocales versus subspaces. There is a difficulty that we will face: contrarily to what happens in the pointfree setting of frames and locales, where sublocales of a given locale L have a concrete description as subsets of L ([3]), the subobjects in the category of σ-locales and σ-localic maps (that is, the dual category of the category of σ-frames and σ-frame homomorphisms), can only be described as *σ-frame congruences* θ on L, that is, equivalence relations on L satisfying the congruence properties

$$(x,y),(x',y') \in \theta \;\Rightarrow\; (x\wedge x', y\wedge y') \in \theta,$$

$$(x_a,y_a) \in \theta \;(a \in A, A = \text{countable set}) \;\Rightarrow\; \Big(\bigvee_{a\in A} x_a, \bigvee_{a\in A} y_a\Big) \in \theta.$$

This is a remarkable difference between the categories of σ-locales and locales.

In this talk, given a σ-space X (that is, a set X equipped with a collection of open sets $\mathcal{O}(X) \subseteq \mathcal{P}(X)$ closed under finite meets and countable joins) and its σ-complete lattice of open sets $\mathcal{O}(X)$, we will focus on the congruences on the σ-frame $\mathcal{O}(X)$ that represent the σ-subspaces of X, referred to as *induced congruences*. We will show that when X is a T_D σ-space, there is a bijection between the σ-subspaces of X and the congruences induced by them.

Let X be a σ-space. From the well-known dual adjunction between the category of σ-spaces and σ-continuous maps and the category of σ-frames and σ-frame homomorphisms,

$$\begin{tikzcd} \sigma\mathsf{Top} \arrow[r, bend left, "\mathcal{O}"] \arrow[r, phantom, "\perp"] & \sigma\mathsf{Frm} \arrow[l, bend left, "\mathsf{Pt}"] \end{tikzcd},$$

one sees that given a σ-subspace $Y \subseteq X$ (with the induced subspace σ-topology) and the inclusion $j\colon Y \hookrightarrow X$, the congruence

$$\theta_{S_Y} := \{(U,V) \in \mathcal{O}(X) \times \mathcal{O}(X) \mid U \cap Y = V \cap Y\}$$

represents an isomorphic imprint of $\mathcal{O}(Y)$ in $\mathcal{O}(X)$. We call it the *congruence induced by the σ-subspace Y.*

We say that a σ-space satisfies the axiom T_D if for any $x \in X$, there is $U_x \in \mathcal{O}(X)$ such that $x \in U_x$ and $U_x \setminus \{x\}$ is still in $\mathcal{O}(X)$. We show that the representation

$$\pi\colon Y \mapsto \theta_Y$$

is one-to-one whenever X is a T_D σ-space:

Proposition. *For a σ-space X, the map $\pi\colon \mathcal{P}(X) \to \mathcal{C}(\mathcal{O}(X))$ from the powerset of X to the congruence lattice of $\mathcal{O}(X)$ is one-to-one if and only if X is T_D. Moreover, it takes arbitrary joins to arbitrary meets but not finite meets to finite joins.*

We will conclude, moreover, still under axiom T_D, that

$$\pi(\mathcal{P}(X)) \subseteq \mathcal{C}_b(\mathcal{O}(X)),$$

where $\mathcal{C}_b(\mathcal{O}(X))$ denotes the subset of $\mathcal{C}(\mathcal{O}(X))$ consisting of all meets of complemented congruences.

We will finish with the remark that imposing a σ-space to be T_D is not too strong, as it encompasses most of the measurable spaces of importance in measure theory, such as euclidean spaces $\mathbb{R}^n$, separable metric spaces or any T_1-space with a countable basis.

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Local finiteness in varieties of MS4-algebras

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An S4-algebra (or closure algebra) is a pair $(B, \Diamond)$ where B is a Boolean algebra and $\Diamond$ is a unary S4-operator on B (a closure operator). S4-algebras provide semantics for the well-known modal logic S4 ([9, 3]). A classical result of Segerberg and Maksimova [10, 7] gives a criterion characterizing when a variety of S4-algebras is locally finite. Via Jónsson–Tarski duality, the variety **S4** of S4-algebras is dually equivalent to a category of descriptive frames (X, R), where X is a Stone space and R is a continuous quasi-order on X. It is then meaningful to speak of the *depth* of an S4-algebra, meaning the longest length of a proper R-chain in its dual frame. We say a variety $\mathbf{V} \subseteq \mathbf{S4}$ has depth $\leq n$ if the depth of each algebra from $\mathbf{V}$ is at most n. Considering the well-known family of formulas (see, e.g., [3, p. 81])

$$P_1 = \Diamond\Box q_1 \to \Box q_1 \qquad P_n = \Diamond(\Box q_n \wedge \neg P_{n-1}) \to \Box q_n$$

we have, for a variety $\mathbf{V} \subseteq \mathbf{S4}$,

1. $\mathbf{V} \models P_n$ iff the depth of $\mathbf{V}$ is $\leq n$ (see, e.g., [3, Prop. 3.44])
2. $\mathbf{V}$ is locally finite iff $\mathbf{V} \models P_n$ for some n (Segerberg–Maksimova).

Thus the locally finite varieties of S4-algebras are precisely those of finite depth. In addition, there is a least subvariety of S4 of infinite depth: the variety **Grz.3** generated by the algebra whose dual space is an infinite descending chain. As a consequence, we may effectively decide if a given variety $\mathbf{V} \subseteq \mathbf{S4}$ is locally finite by determining whether $\mathbf{V} \supseteq \mathbf{Grz.3}$.

An MS4-algebra is a tuple $(B, \Diamond, \exists)$ where $\Diamond$ is an S4-operator and $\exists$ is an S5-operator. Their dual frames are tuples (X, R, E) where X is a Stone space, R is a continuous quasiorder on X, and E is a continuous equivalence relation on X. In the same manner as S4-algebras, we may speak of the depth of an S4-algebra to refer to the longest length of an R-chain in its dual frame. MS4-algebras provide semantics for the modal logic MS4, which may be understood as axiomatizing the one-variable fragment of predicate S4 (QS4); see [4]. In light of this, it is natural to investigate to what extent the Segerberg–Maksimova theorem generalizes to this setting. We give an overview of several results in this direction that are explored in [2].

In the way of positive results, we identify the largest semisimple subvariety of **MS4**, denoted $\mathbf{MS4_S}$, which contains two well-known subvarieties corresponding to $\mathsf{S4}_u$ (S4 extended with the universal modality) and $\mathsf{S5}^2$ (the *product* of S5 with itself, also known as the variety of diagonal-free cylindric algebras of dimension two, see e.g. [5]). We demonstrate that a direct generalization of the Segerberg–Maksimova theorem holds for a family of varieties containing $\mathbf{S4}_u$.

On the other hand, it was known (see, e.g., [6]) that the variety $\mathbf{S5}^2$, which is precisely the variety of MS4-algebras of depth-1, is not locally finite; hence the Segerberg–Maksimova theorem does not generalize directly to **MS4**. We demonstrate that, in fact, characterizing local finiteness in **MS4** is at least as hard as the corresponding problem for $\mathbf{S5}_2$, which remains

*Speaker

wide-open. Here $\mathbf{S5}_2$ corresponds to the *fusion* $\mathsf{S5} * \mathsf{S5}$ – the bimodal logic of two (unrelated) $\mathsf{S5}$ modalities (see, e.g., [5]). We establish this by giving a translation T from subvarieties of $\mathbf{S5}_2$ to subvarieties of $\mathbf{MS4_S} + P_2$ that preserves and reflects local finiteness (i.e., $\mathbf{V}$ is locally finite iff $T(\mathbf{V})$ is locally finite). So already in semisimple subvarieties of depth-2, characterizing local finiteness is difficult.

Finally, we discuss another notable subvariety of $\mathbf{MS4}$, denoted $\mathbf{M^+S4}$. Casari's predicate formula

$$\mathsf{Cas} := \forall x((P(x) \to \forall y P(y)) \to \forall y P(y)) \to \forall x P(x)$$

is well-known in the study of intermediate predicate logic (see, e.g., [8]). In [1] it is shown that the monadic version of Casari's formula is necessary to obtain a faithful provability interpretation of monadic intuitionistic logic. $\mathbf{M^+S4}$ is the subvariety of $\mathbf{MS4}$ obtained by asserting the Gödel translation of this formula, which is then natural to study. Preliminary results indicate that the variety $\mathbf{M^+S4}$ has a much more manageable characterization of local finiteness.

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The maximal spectrum of d-elements is not always Hausdorff

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The d-ideals play an important role in the study of Riesz spaces (see, e.g., [3]). They are exactly the fixpoints of a nucleus on the arithmetic frame of all ideals of a Riesz space. Martinez and Zenk [4] initiated a general study of this nucleus on an arbitrary arithmetic frame. They coined it as the *d-nucleus*. The d-nucleus and its corresponding sublocale were further studied by Bhattacharjee [2], who initiated the study of the spectrum of maximal d-elements. This spectrum is always a locally compact T_1-space, but the question of whether it is Hausdorff was left open.

The aim of this talk is to solve this question in the negative, as well as to give a characterization of when the spectrum is Hausdorff. Our main tool is Priestley duality for the category of bounded distributive lattices [5, 6], and especially its restriction to the category of frames [7, 8]. More specifically, we will utilize Priestley duality for arithmetic frames described in [1].

Let L be an arithmetic frame. For $a \in L$, we write a^* for the pseudocomplement of a in L and define the d-nucleus $d : L \to L$ by

$$da = \bigvee \{k^{**} \mid k \text{ is compact and } k \leq a\}.$$

Let L_d be the sublocale of L of the d-fixpoints. We write X for the Priestley space of L and X_d for the Priestley space of L_d. (Note that $X_d \subseteq X$.)

Let Y be the *localic part* of X (the space of points of L). The localic part of X_d is given by $Y_d = X_d \cap Y$. Since $\mathrm{cl}(Y_d) = X_d$, it is especially important to understand the localic part of X_d. It turns out that $y \in Y_d$ iff y is a relatively maximal localic point of X in the following sense:

Lemma 1. *$y \in Y_d$ iff y is the greatest localic point below a maximal point of X.*

Let $\max(L_d)$ be the spectrum of maximal d-elements [2]. The above lemma gives us means to identify $\max(L_d)$ inside X. of Y. In fact, it is the set Let $\min(Y_d)$ be the set of minimal localic points of X_d.

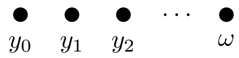
Theorem 2. $\max(L_d)$ *is homeomorphic to* $\min(Y_d)$.

We produce an example of the Priestley space X of an arithmetic frame L such that $\min(Y_d)$ is not Hausdorff. The strategy is to construct a space where $\min(Y_d)$ is homeomorphic to the natural numbers with the cofinite topology. We achieve this as follows. Take the disjoint union of the Stone-Cêch compactification

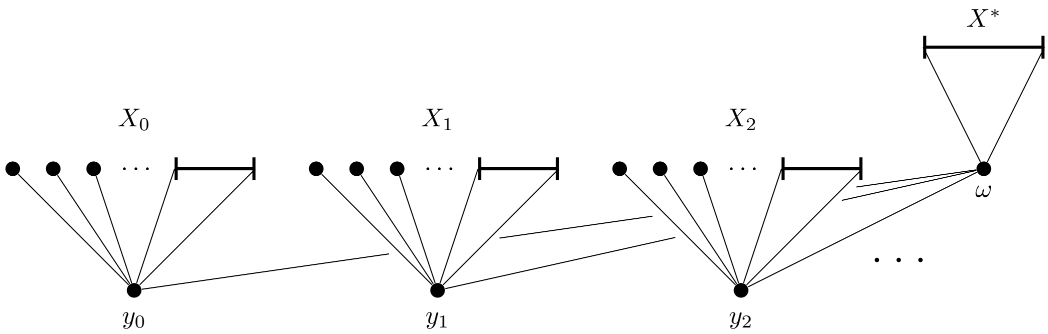
$\beta\mathbb{N} =$ 

[*] Speaker.

and the one-point compactification



of the natural numbers. Then partition $\beta\mathbb{N} = (\bigcup X_i) \cup X^*$ into infinitely many copies X_i of $\beta\mathbb{N}$ and a subset $X^* \subseteq \mathbb{N}^*$. Equipped with the order in the diagram below, we obtain the Priestley space of an arithmetic frame such that $\min(Y_d) = \{y_0, y_1, \dots\}$ is the desired non-Hausdorff space.



Corollary 3. *There are arithmetic frames L such that* $\max(L_d)$ *is not Hausdorff.*

It is worth pointing out that $\max(L_d)$ in the above example is not even sober (recall that a topological space is *sober* if each irreducible closed set is the closure of a unique point). In general, sobriety is strictly weaker than Hausdorffness (i.e., every Hausdorff space is sober, but not vice versa). However, in the case of $\min(Y_d)$, sobriety and Hausdorffness become equivalent properties, thus yielding our characterization:

Theorem 4. $\min(Y_d)$ *is Hausdorff iff* $\min(Y_d)$ *is sober.*

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McKinsey-Tarski Algebras

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In this talk we report on our findings in [3, 4], where an alternate pointfree approach to topology was developed, based on the work of McKinsey and Tarski [10]. We introduce the category **MT** of McKinsey-Tarski algebras and show that it provides a faithful generalization of both **Top** (the category of topological spaces) and **Frm** (the category of frames).

Definition 1.

(1) A *McKinsey-Tarski algebra* (or *MT-algebra* for short) is a pair $M = (B, \Box)$, where B is a complete Boolean algebra and $\Box$ is an interior operator on B (that is, $\Box$ satisfies the Kuratowski axioms $\Box 1 = 1$, $\Box(a \wedge b) = \Box a \wedge \Box b$, $\Box a \leq a$, and $\Box a \leq \Box\Box a$).

(2) An *MT-morphism* between MT-algebras M and N is a complete Boolean homomorphism $h : M \to N$ such that $h(\Box_M a) \leq \Box_N h(a)$ for each $a \in M$.

(3) Let **MT** be the category of MT-algebras and MT-morphisms.

Remark 2.

(1) The study of interior algebras was initiated by McKinsey and Tarski [10]. Interior algebras play an important role in modal logic as they are algebraic models of the well-known modal system **S4** (see, e.g., [11, 5]). MT-algebras are nothing more but complete interior algebras.

(2) MT-morphisms are not homomorphisms of interior algebras, but it is the inequality condition in the above definition that provides a faithful generalization of continuous maps (see [2, 6]). Such morphisms are known as stable homomorphisms (see [1]).

Connection between MT and Frm: Let $M \in$ **MT**. Call an element $a \in M$ *open* if $a = \Box a$. Let $\mathrm{O}(M)$ be the collection of open elements of M. Then $\mathrm{O}(M) \in$ **Frm** and this correspondence extends to a functor O : **MT** $\to$ **Frm**. It is a consequence of Funayama's theorem that O : **MT** $\to$ **Frm** is essentially surjective. However, this does **not** give rise to a functor from **Frm** to **MT**.

Connection to Top: Canonical examples of MT-algebras come from topological spaces. For each $X \in$ **Top**, we have that $(\mathcal{P}(X), \mathrm{int}) \in$ **MT**, and this correspondence gives rise to a contravariant functor $\mathcal{P}$: **Top** $\to$ **MT**. Its contravariant adjoint is given by the functor at : **MT** $\to$ **Top** which maps each MT-algebra M to the space $at(M)$ of atoms equipped with the topology $\eta[\mathrm{O}(M)]$, where $\eta(a) = \{x \in at(M) \mid x \leq a\}$. This gives rise to the contravariant adjunction $(\mathcal{P}, at)$, which restricts to a dual equivalence between **Top** and the reflective subcategory of **MT** consisting of atomic MT-algebras.

Separation axioms in MT-algebras: We generalize the well-known separation axioms for topological spaces and frames to MT-algebras by describing them in terms of the embedding $\mathrm{O}(M) \hookrightarrow M$.

*Speaker.

Sobriety and local compactness: We derive an analogue of the Hofmann-Mislove theorem [8] for sober MT-algebras. Utilizing this result, we establish the MT counterparts of Hofmann-Lawson duality [7] between locally compact frames and locally compact sober spaces and Isbell duality [9] between compact regular frames and compact Hausdorff spaces.

Stone duality: The celebrated Stone duality establishes that the category **BA** of boolean algebras is dually equivalent to the category **Stone** of Stone spaces. We define the category **StoneMT** of Stone MT-algebras and show that it is equivalent to both **BA** and the category **StoneFrm** of Stone frames. The equivalence between **StoneFrm** and **StoneMT** is obtained by restricting O. The equivalence between **StoneMT** and **BA** is established as follows.

The functor Clp : **StoneMT** $\to$ **BA** associates with each MT-algebra M the boolean algebra of clopen elements of M. A quasi-inverse of Clp : **StoneMT** $\to$ **BA** is the functor $(-)^\sigma$: **BA** $\to$ **StoneMT** which associates with each boolean algebra B the Stone MT-algebra $M = (B^\sigma, \Box)$, where B^σ is the canonical extension of B and $\Box : B^\sigma \to B^\sigma$ is defined by $\Box x = \bigvee\{b \in B \mid b \leq x\}$.

$$\mathbf{StoneFrm} \xleftarrow{\quad \mathrm{O} \quad} \mathbf{StoneMT} \underset{\mathrm{Clp}}{\overset{(-)^\sigma}{\rightleftarrows}} \mathbf{BA}$$

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Embeddings of metric Boolean algebras in $\mathbb{R}^N$

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A metric Boolean algebra (see e.g. [1, 2, 3]) consists of a Boolean algebra $\mathbf{A}$, equipped with a strictly positive (finitely-additive) probability measure[1] $m\colon \mathbf{A} \to [0,1]$, which makes $(\mathbf{A}, d_m)$ a metric space, where the distance between any two points $a, b \in A$ is defined as:

$$d_m(a,b) \coloneqq m(a \triangle b) = m((a \wedge b') \vee (a' \wedge b)).$$

From a geometrical point of view, it is natural to wonder under which conditions a metric Boolean algebra $(\mathbf{A}, d_m)$, or some of its relevant subspaces, can be isometrically embedded in $\mathbb{R}^N$ (equipped with the Euclidean distance), for a given positive integer N. Actually, for $|A| > 2$, there is no such embedding. However, under the assumption that $\mathbf{A}$ is finite (or, more generally, atomic), it makes sense to restrict the question to the subspace $\mathsf{At}(\mathbf{A})$ of its atoms.

A classical result by Morgan [5] states that a metric space (X,d) embeds in $\mathbb{R}^N$ if and only if it is flat and has dimension less or equal to N, where (X,d) is flat if the determinant of the matrix $M(\vec{x}_n)$, whose generic entry is $M_{ij} = \frac{1}{2}(d(x_0,x_i)^2 + d(x_0,x_j)^2 - d(x_i,x_j)^2)$, is non-negative for every n-simplex (namely every choice of $n+1$ points $\vec{x}_n = \{x_0, \ldots, x_n\}$ in X) and the dimension of (X,d) is the greatest N (if exists) such that there exists a N-simplex with positive determinant.

Given a finite metric Boolean algebra $\mathbf{A}$ with $\mathsf{At}(\mathbf{A}) = \{a_0, a_1, \ldots, a_k\}$, it is easily checked that the matrix $M(\vec{x}_n) = \{M_{ij}\}$, $2 \leq n \leq k$ (introduced in Morgan's theorem) has generic entry

$$M_{ij} = (x_0 + x_i)^2 \delta_{ij} + (x_0^2 + x_0 x_1 + x_0 x_j - x_i x_j)(1 - \delta_{ij}),$$

where $x_\alpha = m(a_\alpha)$ (thus $x_\alpha > 0$, for every $\alpha \in \{0, 1, \ldots, k\}$). Therefore the form of the determinant can be simplified according to the following.

Lemma 1. *Let $M(\vec{x}_n)$, $2 \leq n \leq k$ be the matrix associated to a finite metric atomic Boolean algebra $\mathbf{A}$ with $k+1$ atoms. Then*

$$\det(M(\vec{x}_n)) = 2^{n-1} \left[\left(\sum_{\alpha=0}^{n} x_0 \cdot \cdots \cdot \hat{x}_\alpha \cdot \cdots \cdot x_n \right)^2 - (n-1) \left(\sum_{\alpha=0}^{n} x_0^2 \cdot \cdots \cdot \hat{x}_\alpha^2 \cdots \cdot x_n^2 \right) \right],$$

where $\hat{x}_i$ means that x_i has to be omitted.

It follows, for instance, that the space $(\mathsf{At}(\mathbf{A}), d_m)$ of the $k+1$ atoms of a finite metric Boolean algebra such that $m(a_i) = \frac{1}{k+1}$ (for every $a_i \in \mathrm{At}(\mathbf{A})$) embeds in $\mathbb{R}^k$ with the Euclidean metric and that $\det(M(\vec{x}_2)) > 0$.

[1] Recall that a *strictly positive* (finitely additive) probability measure over a Boolean algebra $\mathbf{A}$ is a map $m\colon \mathbf{A} \to [0,1]$ such that:

1. $m(\perp) = 1$,
2. $m(a \vee b) = m(a) + m(b)$, for every $a, b \in A$ such that $a \wedge b = \perp$,
3. $m(a) > 0$, for every $a \in A$, $a \neq \perp$.

Upon indicating by $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ the space of the (finitely additive) probability measures m such that $(\mathrm{At}(\mathbf{A}), d_m)$ admits an isometric embedding into some Euclidean space $\mathbf{R}^N$, in virtue of Morgan's theorem one has

$$\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A})) = \bigcap_{n=3}^{k} C_n \cap \Pi_k,$$

where $C_n = \{\vec{x} \in \mathbb{R}_+^{k+1} \mid \det M(\vec{x}_n) \geq 0\}$, with $3 \leq n \leq k$ and Π_k is the interior of the standard k-simplex (or probability simplex) of $\mathbb{R}^{k+1}$, namely

$$\Pi_k = \{\vec{x} \in (0,1)^{k+1} \mid \sum_{\alpha=0}^{k} x_\alpha = 1\}.$$

We are interesting in solving the following.
Problem. Study the topology of $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ with the topology induced by $(0,1)^{k+1} \subset \mathbb{R}_+^{k+1}$.

In order to get a solution, we first analyze the topology of C_n.

Lemma 2. *For each $3 \leq n \leq k$, the space $C_n \cong H_n \times \mathbb{R}_+^{k-n}$ where H_n is a solid half-hypercone in $\mathbb{R}_+^{n+1}$.*

The solution to the above presented problem is given by the following.

Theorem 3. *Let $k \geq 3$. Then:*

1. *$\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ is contractible.*
2. *$\mathcal{M}(\mathrm{At}(\mathbf{A})) \setminus \mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ is simply-connected (not contractible).*

In the final part of the talk, we will draw some considerations on the significance of our results for probability theory and on their possibile extensions to the case of infinite (non-atomic) Boolean algebras.

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Modal Weak Kleene Logics

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The basic weak Kleene logics B_3 and PWK (standing for Bochvar logic and Paraconsistent weak Kleene, respectively) can be introduced as three-valued logics characterized by an infectious non-classical value. While these logics have interesting syntactic properties as examples of variable inclusion logics (see [2]), from the perspective of (abstract) algebraic logic they are very weakly connected to their algebraic counterparts.

Once the language of these logics - that is in the type of classical logic - is enriched with an unary operator J_2, whose intuitive reading is "it is (classically) true that", the resulting expansions - also known as *external* Kleene logics - gain strong algebraic properties. The operator J_2 allows to define two other operators, and their semantics is explained by the following tables:

φ	$J_2\varphi$
1	1
$1/2$	0
0	0

φ	$J_0\varphi$
1	0
$1/2$	0
0	1

φ	$J_1\varphi$
1	0
$1/2$	1
0	0

The three mentioned connectives form the so-called external operators, in the sense that each formula in which every propositional variable falls under the scope of those operators behaves entirely classically. Therefore the expansions of B_3 and PWK with J_2 are called, respectively, Bochvar external logic B_e and external paraconsistent weak Kleene logic PWK_e.

The addition of J_2 is enough to restore the algebraic connection that basic, non-external Kleene logics lacked: both B_e and PWK_e are algebraizable, as proved, respectively, in [1] and [4]. These logics share the quasi-variety of Bochvar algebras BCA as their equivalent algebraic semantics. BCA was introduced in [5] and has been recently studied in [3], which provided a representation theorem of the J_2-free reduct of Bochvar algebras in terms of Płonka sums of Boolean algebras (plus additional operations). Płonka sums [6, 7] are an algebraic construction which allows to construct a new algebra starting from a semilattice direct system +of similar algebras. This tool has revealed its efficacy in the algebraic study of algebras connected with weak Kleene logics and, more in general, for the logics of variable inclusion [2].

In this work we present a study started in [4] on modal weak Kleene external logics. The language of weak Kleene external logics can be expanded with a modal operator $\Box$, whose intended meaning is that of standard alethic modal logic, a task first undertaken by Segerberg in [8] with a less general scope. Our work is divided into two parts: the first focuses on Kripke-style semantics, the other on algebraic semantics for modal weak Kleene external systems. We introduce the logics $\mathsf{B}_\mathrm{e}^\Box$ and $\mathrm{PWK}_\mathrm{e}^\Box$, respectively modal Bochvar external logic and modal external PWK. The reading of the $\Box$ modality differs between the two systems, according to the underlying propositional logic. Using a possible world interpretation, the intuitive reading of $\Box\varphi$ is "φ is true at every accessible world" in $\mathsf{B}_\mathrm{e}^\Box$, and "$\varphi$ is non-false at every accessible world" in $\mathrm{PWK}_\mathrm{e}^\Box$. The logics has been axiomatized and a complete Kripke-style semantics is provided for both. The systems are also decidable and easy to extend axiomatically, obtaining completeness results w.r.t. classes of frames characterized by well-known properties.

In the algebraic part we introduce the global versions of the local modal logics $\mathsf{B}_{\mathrm{e}}^{\Box}$ and $\mathrm{PWK}_{\mathrm{e}}^{\Box}$, respectively $g\mathsf{B}_{\mathrm{e}}^{\Box}$ and $g\mathrm{PWK}_{\mathrm{e}}^{\Box}$. We present a study of the algebraic counterparts of these logics, first introducing the quasi-variety of modal Bochvar algebras MBCA, and then identifying the two subclasses $\mathsf{MBCA_B}$ and $\mathsf{MBCA_H}$, which are the equivalent algebraic semantics of $g\mathsf{B}_{\mathrm{e}}^{\Box}$ and $g\mathrm{PWK}_{\mathrm{e}}^{\Box}$, respectively. The choice to move from local to global logics is motivated by the failure of algebraizability for local modal systems, algebraizability that is recovered once we consider their global versions. Building upon the results obtained in [3], we prove a representation theorem for $\mathsf{MBCA_B}$ and $\mathsf{MBCA_H}$, which states that the J_2-free reduct of a modal Bochvar algebra belonging to these classes is a particular Płonka sum of Boolean algebras with operators. We show how certain relative sub-varieties of these classes correspond to standard extensions of the basic modal logics $g\mathsf{B}_{\mathrm{e}}^{\Box}$ and $g\mathrm{PWK}_{\mathrm{e}}^{\Box}$ which are characterized by well-known frame properties from the side of their Kripke semantics.

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Kites and pseudo MV-algebras

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We deal with variants of a special construction of certain algebras called *kites*, naturally associated with a noncommutative generalisation of BL-algebras known as *pseudo BL-algebras*. We focus on *perfect pseudo MV-algebras* generalising previous results by DiNola, Dvurečenskij and Tsinakis. The following varieties of algebras will also play an important role: LG – lattice-ordered groups, $\mathsf{CanIGMV}$ – cancellative integral generalised MV-algebras, $\Psi\mathsf{MV}$ – pseudo MV-algebras.

Definition 1. *An* $\mathsf{FL_w}$*-algebra* $\mathbf{A}$ *is said to be* perfect *if there is a homomorphism* $h_{\mathbf{A}}\colon \mathbf{A} \to \mathbf{2}$ *such that for any* $x \in h_{\mathbf{A}}^{-1}(0)$ *and any* $y \in h_{\mathbf{A}}^{-1}(1)$ *the inequality* $x \leq y$ *holds.*

We say that a variety $\mathcal{V}$ of $\mathsf{FL_w}$-algebras is *perfectly generated* if it is generated by its perfect members. If a perfectly generated variety $\mathcal{V}$ is a subvariety of some larger variety, and that larger variety has a well established name, say NN, we write PNN for the perfectly generated subvariety of NN.

Let $\mathbf{A}$ be an FL-algebra, and $a, b \in A$. The *left conjugate* of $a \in A$ by $b \in A$ is the element $\lambda_b(a) := (b \setminus ab) \wedge 1$ and the *right conjugate* is $\rho_b(a) := (ba \,/\, b) \wedge 1$. A *conjugation polynomial* $\boldsymbol{\alpha}$ over $\mathbf{A}$ is any unary polynomial $(\gamma_{a_1} \circ \gamma_{a_2} \circ \cdots \circ \gamma_{a_n})(x)$ where $\gamma \in \{\lambda, \rho\}$ and $a_i \in A$ for $1 \leq i \leq n$. We write $\mathrm{cPol}(\mathbf{A})$ for the set of all conjugation polynomials over $\mathbf{A}$. For an element $u \in A$, an *iterated conjugate* of u is $\boldsymbol{\alpha}(u)$ for some $\boldsymbol{\alpha} \in \mathrm{cPol}(\mathbf{A})$.

Theorem 1. *A subvariety* $\mathcal{V}$ *of* $\mathsf{FL_w}$ *is perfectly generated if and only if* $\mathcal{V}$ *is nontrivial and satisfies the following identities:*

$$\boldsymbol{\alpha}(x \,/\, x^-) \vee \boldsymbol{\beta}(x^- \,/\, x) = 1, \tag{1}$$

$$\boldsymbol{\alpha}((x \vee x^-) \cdot (y \vee y^-))^- \leq \boldsymbol{\alpha}((x \vee x^-) \cdot (y \vee y^-)), \tag{2}$$

$$x \wedge x^- \leq y \vee y^- \tag{3}$$

for every $\mathbf{A} \in \mathcal{V}$ *and all* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{cPol}(\mathbf{A})$.

A natural generalisation of MV-algebras is the variety $\Psi\mathsf{MV}$ of *pseudo MV-algebras.* By a result of Dvurečenskij, pseudo MV-algebras are categorically equivalent to the class of lattice-ordered groups with strong unit, so in a good sense they are the largest possible generalisation of MV-algebras for which a Mundici-type categorical equivalence holds. For pseudo MV-algebras, we define a natural generalisation of the kite construction.

Definition 2. *Let* $\mathbf{L}$ *be an* ℓ*-group and* $\lambda\colon \mathbf{L} \to \mathbf{L}$ *be an automorphism. We define the algebra*

$$\mathcal{K}(\mathbf{L}, \lambda) := (L^- \uplus L^+; \wedge, \vee, \odot, \setminus, /, 0, 1)$$

where $L^- \uplus L^+$ *is a disjoint union,* $0 := e \in L^+$*,* $1 := e \in L^-$*, and the other operations are given by*

$$x \wedge y := \begin{cases} x \wedge y & \text{if } x, y \in L^-, \\ x & \text{if } x \in L^+, y \in L^- \\ y & \text{if } x \in L^-, y \in L^+, \\ x \wedge y & \text{if } x, y \in L^+, \end{cases} \qquad x \vee y := \begin{cases} x \vee y & \text{if } x, y \in L^-, \\ y & \text{if } x \in L^+, y \in L^- \\ x & \text{if } x \in L^-, y \in L^+, \\ x \vee y & \text{if } x, y \in L^+, \end{cases}$$

$$x \odot y := \begin{cases} x \cdot y & \text{if } x, y \in L^-, \\ \lambda(x) \cdot y \vee 0 & \text{if } x \in L^-, y \in L^+ \\ x \cdot y \vee 0 & \text{if } x \in L^+, y \in L^-, \\ 0 & \text{if } x, y \in L^+, \end{cases}$$

$$x \setminus y := \begin{cases} x^{-1} \cdot y \wedge 1 & \text{if } x, y \in L^-, \\ 1 & \text{if } x \in L^+, y \in L^- \\ \lambda(x)^{-1} \cdot y \vee 0 & \text{if } x \in L^-, y \in L^+, \\ x^{-1} \cdot y \wedge 1 & \text{if } x, y \in L^+, \end{cases} \qquad y \,/\, x := \begin{cases} y \cdot x^{-1} \wedge 1 & \text{if } x, y \in L^-, \\ 1 & \text{if } x \in L^+, y \in L^- \\ y \cdot x^{-1} \vee 0 & \text{if } x \in L^-, y \in L^+, \\ \lambda^{-1}(y \cdot x^{-1}) \wedge 1 & \text{if } x, y \in L^+, \end{cases}$$

Theorem 2. *Let* $\mathbf{A}$ *be a perfect pseudo MV-algebra. Then* $\mathbf{A} \cong \mathcal{K}(\ell(\mathbf{F_A}), \ell^{\approx})$*, where* $\ell^{\approx}$ *is the automorphism induced by the term operation* $x^{\approx} := 0\backslash(0\backslash x)$*.*

The next theorem generalises some results by Di Nola, Dvurečenskij and Tsinakis.

Theorem 3. *The category* $\mathsf{pf\Psi MV}$ *of perfect pseudo MV-algebras is equivalent to the category of lattice-ordered groups with a distinguished automorphism.*

We also obtain a characterisation of varieties generated by kites, and a description of the lattice of such varieties. For a variety $\mathcal{V}$ of algebras, we let $\Lambda(\mathcal{V})$ stand for the lattice of subvarieties of $\mathcal{V}$. If the poset of nontrivial subvarieties of $\mathcal{V}$ is also a lattice, we let $\Lambda^+(\mathcal{V})$ stand for that lattice. We denote by $\mathbb{D}$ the *divisibility lattice*, that is, $\mathbb{N}$ ordered by divisibility. The parameter n and $dim(\mathcal{V})$ below refer to a notion of *dimension* of a variety, which we leave undefined here for lack of space.

Definition 3. *We define two pairs of maps*

$$\psi\colon \Lambda(\mathsf{P\Psi MV}) \to \Lambda(\mathsf{CanlGMV}), \text{ where } \psi(\mathcal{V}) = V\{\mathbf{F_A} : \mathbf{A} \in \mathcal{V}_{pf}\},$$
$$\Psi\colon \Lambda(\mathsf{P\Psi MV}) \to \Lambda(\mathsf{CanlGMV}) \times \mathbb{D}, \text{ where } \Psi(\mathcal{V}) = (\psi(\mathcal{V}),\ dim(\mathcal{V})),$$

for any $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$ *and*

$$\delta\colon \Lambda(\mathsf{CanlGMV}) \to \Lambda(\mathsf{P\Psi MV}), \text{ where } \delta(\mathcal{V}) = V\{\mathbf{A} \in \mathsf{pf\Psi MV} : \mathbf{F_A} \in \mathcal{V}\},$$
$$\Delta\colon \Lambda(\mathsf{CanlGMV}) \times \mathbb{D} \to \Lambda(\mathsf{P\Psi MV}), \text{ where } \Delta(\mathcal{V}, n) = \delta(\mathcal{V}) \cap \mathsf{P\Psi MV}_n,$$

for any $\mathcal{V} \in \Lambda(\mathsf{CanlGMV})$ *and* $n \in \mathbb{D}$*.*

Theorem 4. *Let* $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$*. The following are equivalent.*

1. $\mathcal{V}$ *is generated by kites.*
2. $\mathcal{V} = \Delta\Psi(\mathcal{V})$*.*
3. $\mathcal{V} = \Delta(\mathcal{W}, n)$ *for some* $\mathcal{W} \in \Lambda(\mathsf{CanlGMV})$ *and some* $n \in \mathbb{D}$*.*

Theorem 5. *Let* $\mathbb{K}$ *be the lattice of subvarieties of* $\mathsf{P\Psi MV}$ *generated by kites.*

$$\mathbb{K} \cong \mathbf{1} \oplus \big(\Lambda^+(\mathsf{CanlGMV}) \times \mathbb{D}\big) \cong \mathbf{1} \oplus \big(\Lambda^+(\mathsf{LG}) \times \mathbb{D}\big)$$

where $\mathbf{1}$ *is the trivial lattice and* $\oplus$ *is the operation of ordinal sum.*

An algebraic semantics for possibilistic finite-valued Łukasiewicz logic

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In the present work, based on the ideas of [1], we analyse an algebraic semantics for a many-valued modal logical system based on the n-valued Łukasiewicz logic $\Lambda(\mathbf{Ł}_n)$. We extend $\Lambda(\mathbf{Ł}_n)$ (for each fixed natural n) to a modal system, by adding a unary operator $\Box$ to the original many-valued propositional language. Our approach strongly relies on the fact that the system $\Lambda(\mathbf{Ł}_n)$ has as an algebraic semantics the subvariety of MV-algebras generated by the n-elements MV-chain $\mathbf{Ł}_n$, that is the algebra with universe $\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$, and main operations $x \to y = \min\{1, 1-x+y\}$, $\neg x = 1-x$.

To achieve our aim, we work with a propositional modal language formed by a numerable set of variables Var and the connectives $\langle \to, \neg, \Box, 0\rangle$, thus the set $Form_\Box$ of formulas in this language is built as usual. An $\mathbf{Ł}_n$-valued *possibilistic frame* $\langle W, \pi\rangle$ is given by a non-empty set of worlds W and a function $\pi : W \to \mathbf{Ł}_n$ (called a normalized possibility distribution over W) such that $\bigvee_{w\in W} \pi(w) = 1$. An *$\mathbf{Ł}_n$-valued possibilistic model* is a 3-tuple $M = \langle W, \pi, e\rangle$ where $\langle W, \pi\rangle$ is an $\mathbf{Ł}_n$-valued possibilistic frame and e is a map, called *valuation*, assigning to each propositional variable in Var and each possible world in W an element of $\mathbf{Ł}_n$ (i.e., $e : Var \times W \longrightarrow \mathbf{Ł}_n$).

If $M = \langle W, \pi, e\rangle$ is a $\mathbf{Ł}_n$-valued possibilistic model, the map e can be uniquely extended to a map, assigning to each formula in $Form_\Box$ and each world in W an element of $\mathbf{Ł}_n$ (i.e., $e : Form_\Box \times W \longrightarrow \mathbf{Ł}_n$) satisfying that:

- e is an algebraic homomorphism in its first component, i.e., for the connectives $\to, \neg, 0$,
- $e(\varphi, w) = \bigwedge\{\pi(w') \to e(\varphi, w') : w' \in W\}$

The logical system that we are trying to characterize generalizes the classical possibilistic logic, and it is the many-valued modal system semantically defined by possibilistic models over $\mathbf{Ł}_n$

Our algebraic approach deals with complex algebras that arise from $\mathrm{Ł}_n$-valued frames. That is: given an $\mathbf{Ł}_n$-valued possibilistic frame $\langle W, \pi\rangle$ we consider the MV-algebra of functions $\mathrm{Ł}_n^W$ and the unary operator $\Box : \mathrm{Ł}_n^W \to \mathrm{Ł}_n^W$ given by $\Box x(i) = \bigwedge\{\pi(j) \to x(j) : j \in W\}$. We study the quasivariety of algebras generated by these complex algebras, and this quasivariety, together with the abstract theory of algebraizable logics immediately provide an axiomatization for the possibilistic many-valued system over $\mathbf{Ł}_n$. From the way that the system is defined, it turns out to be complete with respect to the logic semantically defined by the $\mathbf{Ł}_n$-valued possibilistic frames. So the logical system determined by frames over $Ł_n$ has an algebraic semantics based on MV-algebras.

The present investigation provides a negative answer to a conjecture of P. Hájek posed in his book [2] which intends to generalize the classical setting, where the possibilistic logic coincides with the modal logic $KD45$. We prove that the logic semantically defined by $\mathbf{Ł}_n$-valued possibilistic frames can not be axiomatized by simply requiring the fuzzy analogues of the classical axioms K,D,4 and 5.

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Epimorphisms between finitely generated algebras

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Let K be a class of similar algebras and $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$.

Definition 1. *A homomorphism $f \colon \mathfrak{A} \to \mathfrak{B}$ is an* epimorphism *in K when for every $\mathfrak{C} \in \mathsf{K}$ and every pair of homomorphisms $g, h \colon \mathfrak{B} \to \mathfrak{C}$ it holds that:*

$$g \circ f = h \circ f \text{ implies } g = h.$$

A subalgebra $\mathfrak{A} \leq \mathfrak{B}$ is called epic *in K if the inclusion $i \colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ is an epimorphism in K.*

While every surjective homomorphism is an epimorphism, the converse is not true in general. An example of a nonsurjective epimorphism in the class of rings is the inclusion map from the integers into the rationals (see, e.g., [6]).

Definition 2. *When every epimorphism in K is surjective, we say that K has the* epimorphism surjectivity property *(ES property, for short).*

Our talk will focus on a slightly weaker demand, namely, the *weak epimorphism surjectivity property* (weak ES property, for short), which requires only epimorphisms between finitely generated algebras to be surjective [5]. From a logical standpoint, the interest of the weak ES property is motivated as follows: when a quasivariety K algebraizes a logic $\vdash$, the former has the weak ES property iff the latter has the Beth definability property [1], which intuitively states that whenever an element can be uniquely characterized, then it must be definable by a term.

Our main results facilitate the detection of failures of the weak ES property in a quasivariety K. To this end, we introduced the notion of a *full subalgebra*.

Definition 3. *A subalgebra $\mathfrak{A} \leq \mathfrak{B} \in \mathsf{K}$ is* full *when it is proper, $B = \mathrm{Sg}^{\mathfrak{B}}(A \cup \{b\})$ for some $b \in B$, and for every nonidentity K-congruence θ of $\mathfrak{B}$ there exists $a \in A$ such that $\langle a, b \rangle \in \theta$.*

Using this concept, we obtained the following characterization of the weak ES property, where K_{RFSI} stands for the class of *relatively subdirectly irreducible* (RFSI, for short) members of K.

Theorem 4. *A quasivariety K has the weak ES property iff for every finitely generated $\mathfrak{B} \in \mathsf{K}$ and $\mathfrak{A} \leq \mathfrak{B}$ that is full in K one of the following conditions holds:*

1. *There are two distinct $\theta, \phi \in \mathsf{Con}_{\mathsf{K}}(\mathfrak{B})$ such that $\theta{\restriction}_A = \phi{\restriction}_A$;*

2. *There are two distinct embeddings $g, h \colon \mathfrak{B} \to \mathfrak{C}$ with $\mathfrak{C} \in \mathsf{K}_{\text{RFSI}}$ such that $g{\restriction}_A = h{\restriction}_A$.*

*Speaker.

As a consequence, we obtain a purely algebraic proof of a classical result of Kreisel, stating that every variety of Heyting algebras or implicative semilattices has the weak ES property [7, Thm. 1]. On the other hand, Theorem 4 also paves the way for the following results.

Our first theorem simplifies the task of finding a counterexample to the weak ES property in quasivarieties with a near unanimity term. This includes, for instance, all quasivarieties with a lattice reduct.

Definition 5. *A quasivariety* K *is said to have an* n-ary near unanimity term *for* $n \geq 3$ *when there exists a term* $\varphi(x_1, \ldots, x_n)$ *such that*

$$\mathsf{K} \vDash \varphi(y, x, \ldots, x) \approx \varphi(x, y, x, \ldots, x) \approx \cdots \approx \varphi(x, \ldots, x, y) \approx x.$$

Theorem 6. *A quasivariety* K *with an* n*-ary near unanimity term has the weak ES property iff every finitely generated subdirect product* $\mathfrak{A} \leq \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_{n-1}$*, where* $\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1} \in \mathsf{K}_{\text{RFSI}}$*, lacks subalgebras that are full and epic in* K*.*

The next result gives a useful characterization of the weak ES property in the context of congruence permutable varieties. Notably, these include all varieties with a group reduct.

Theorem 7. *A congruence permutable variety has the weak ES property iff its finitely generated RFSI members lack subalgebras that are full and epic in* K*.*

Similar results for the ES property have been obtained by Campercholi [2, Thms. 18 and 22]. For instance, [2, Thm. 22] states that an arithmetical variety K, whose class of RFSI members is universal has the ES property iff the RFSI members of K lack proper subalgebras that are epic in K. Our methods allow us to prove a similar result for the weak ES property (namely, Theorem 7) under the sole assumption that K is congruence permutable.

Lastly, we provide a result which demonstrates that the weak ES property has a significant impact on the structure theory of quasivarieties.

Theorem 8. *Let* K *be a relatively congruence distributive quasivariety, whose class of RFSI members is closed under nontrivial subalgebras. Then the weak ES property implies that* $\mathbb{V}(\mathsf{K})$ *is arithmetical.*

As a consequence, every filtral variety with the weak ES property is a discriminator variety (see also [3]). The results of this talk have been collected in the manuscript [4].

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Finite Homomorphism Preservation in Many-Valued Logics

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A canonical result in model theory is the Homomorphism Preservation Theorem (h.p.t.) which states that a first-order formula is preserved under homomorphisms on all structures if and only if it is equivalent to an existential-positive formula. It is an example of a preservation theorem, linking a syntactic class of formulas with preservation under a particular kind of map between structures and standardly proved via a compactness argument. Rossman [1] established that the h.p.t. remains valid when restricted to finite structures, yielding the following formulation of the theorem.

Finite Homomorphism Preservation Theorem A first-order sentence of quantifier-rank n is preserved under homomorphisms on finite structures iff it is equivalent in the finite to an existential-positive sentence of quantifier rank $\rho(n)$ (for some explicit function $\rho : \omega \to \omega$). That is, for any first-order sentence ϕ of quantifier rank n, $Mod_{fin}(\phi)$ is closed under homomorphisms iff there is an existential-positive sentence ψ of quantifier rank $\rho(n)$ such that for all finite models $M \models \phi$ iff $M \models \psi$.

This is a significant result in the field of finite model theory. It stands in contrast to other results proved via compactness, including the other preservation theorems where the failure of the compactness also results in the failure of the derived theorem [2]. It is also an important result for the field of constraint satisfaction due to the equivalence of existential-positive formulas and unions of conjunctive queries. Adjacently, Dellunde and Vidal [3] established that a version of the h.p.t. holds for a collection of many-valued models, those defined over a fixed finite MTL-chain.

MTL Finite Homomorphism Preservation Theorem Let $\mathcal{P}$ be a predicate language, A a finite MTL-chain and ϕ a consistent sentence over A. Then ϕ is equivalent over A to an existential-positive sentence iff $Mod^{A}_{fin}(\phi)$ is closed under homomorphisms.

MTL-algebras provide the algebraic semantics for the monoidal t-norm logic MTL, a basic propositional fuzzy logic that encompasses the most well-studied fuzzy logics including Hájek's basic logic BL, Gödel–Dummett logic G and Łukasiewicz logic Ł [5, Chapter 1, Section 2]. Much like Rossman's work, Dellunde and Vidal's investigation is further motivated by the application of models defined over MTL-algebras to *valued* constraint satisfaction problems (VCSP), a generalisation of classical CSP, where constraints are assigned some form of weighting which is optimised for in the solution. This has been effectively modelled by taking the weights as elements of an algebra and utilising the algebraic operations to interpret their combination in a potential solution [4], MTL-algebras providing one example. Our investigation picks up at the meeting point of these two strands. One can extend Rossman's proof of a finite h.p.t. to a very wide collection of many-valued models, which in particular establishes a finite variant to Dellunde and Vidal's result. In fact, we work with more general algebras than MTL-chains, the somewhat artificial class of algebras we refer to as interpreting algebras and we consider the case where we allow our models to be defined over varying interpreting algebras.

Definition An *interpreting algebra* is an algebra A in signature $\mathcal{L} = \langle \wedge, \vee, \&, 1\rangle$ such that:

$\langle A, \wedge, \vee\rangle$ is a distributive lattice; $\langle A, \&, 1\rangle$ is a commutative (abelian) monoid;

$\forall a, b, c \in A, a \leq b$ implies $a \;\&\; c \leq b \;\&\; c$. $\forall a, b \in A, a \vee b \geq 1$ implies $a \geq 1$ or $b \geq 1$.

In the many-valued setting both the notion of homomorphism and existential-positive formulas split into a number of interrelated concepts and this naturally provides a number of possible generalisations of the classical h.p.t. As it turns out, the appropriate variant links *protomorphisms* with *existential-$\wedge$-positive sentences* ($\exists.\wedge$.p).

Definition Let $(A, M), (B, N)$ be $\mathcal{P}$-models. A map $g \colon M \to N$ is a *protomorphism* from (A, M) to (B, N) iff:

- for every $F \in \mathcal{P}$ and $\bar{m} \in M$ $g(F^M(\bar{m})) = F^N(g(\bar{m}))$.
- for every $R \in \mathcal{P}$ and $\bar{m} \in M$ $R^M(\bar{m}) \geq 1$ implies $R^N(g(\bar{m})) \geq 1$.

Let $f \colon A \to B$ and $g \colon M \to N$ be maps. We call the pair $(f, g) \colon (A, M) \to (B, M)$ a *homomorphism* from (A, M) to (B, N) iff f is an algebraic $\mathcal{L}$-homomorphism and g is a protomorphism from (A, M) to (B, N). We write $\to_p$ $(\to)$ to indicate there exists a protomorphism (homomorphism) between two $\mathcal{P}$-models.

Given a predicate language $\mathcal{P}$ and a $\mathcal{P}$-formula ϕ it is said that ϕ is existential-$\wedge$-positive iff ϕ is built using the connectives $\wedge$ and $\vee$ and the existential quantifier $\exists$.

One can easily check by induction that $\exists.\wedge$.p sentences are preserved under protomorphisms. Our strategy for the other direction is to translate between $\mathcal{P}$-models defined over interpreting algebras and a 'classical counterpart' in such a way that the behaviour regarding protomorphisms and $\exists.\wedge$.p-sentences is preserved. The classical translations are presented as a $\mathcal{P}$-model taken over the 2 element Boolean algebra $\{\top, \bot\}$.

Definition Let (A, M) be a $\mathcal{P}$-model over an interpreting algebra A. We define the $\mathcal{P}$-model $(\{\top, \bot\}, M^\top)$, also denoted simply as $M^\top$ as follows:

$$R^{M^\top}(\bar{m}) = \begin{cases} \top & \text{if } R^M(\bar{m}) \geq 1_A \\ \bot & \text{if } R^M(\bar{m}) < 1_A. \end{cases}$$

One can then apply Rossman's results to these objects (viewed as a classical models) before pulling back into the many-valued setting, yielding our many-valued equivalent.

Finite Protomorphism Preservation Theorem Let $\mathcal{P}$ be a predicate language and ϕ a consistent $\mathcal{P}$-sentence. Then ϕ is equivalent in the finite to an $\exists.\wedge$.p-sentence ψ iff $Mod_{fin}(\phi)$ is preserved under protomorphisms.

Moreover, when one restricts to models defined over a fixed algebra, the usual notion of homomorphism collapses with protomorphisms. This lets us freely add it to the equivalence.

Fixed Finite Homomorphism Preservation Theorem Let $\mathcal{P}$ be a predicate language, A an interpreting algebra and ϕ a consistent $\mathcal{P}$ sentence over A in the finite. The following are equivalent:

1. ϕ is equivalent over A in the finite to an $\exists.\wedge$.p sentence ψ, i.e. there is an $\exists.\wedge$.p-sentence $\psi : Mod^A_{fin}(\phi) = Mod^A_{fin}(\psi)$.
2. ϕ is preserved under protomorphisms on A, i.e. $Mod^A_{fin}(\phi)$ is closed under $\to_p$.
3. ϕ is preserved under homomorphisms on A, i.e. $Mod^A_{fin}(\phi)$ is closed under $\to$.

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On assume-guarantee contract algebras

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Contract-based design has emerged as a way to design a wide variety of systems in engineering. Contracts have a legal part and a technical part. Only the technical part will be relevant in what follows. Informally, a contract is a pair (A, G), where A is the set of assumptions under which the system is assumed to operate and G is the set of guarantees that the system must provide. For examples and details see [1].

In [3], an algebraic perspective on assume-guarantee contracts is proposed. This proposal relies heavily on a construction involving Boolean algebras. Given any Boolean algebra $\mathbf{B} = (B, \cap, \cup, \,', 0, 1)$, the set of pairs $(a, b) \in B \times B$ such that $a \cup b = 1$ is taken to be the universe of the assume-guarantee contract algebra associated to $\mathbf{B}$. Let us call $S(B)$ to this set. However, the structures thus proposed lack a clearly prescribed set of basic operations, necessary if we want to see them as a class of algebras in the sense of Universal Algebra.

Many different operations can be defined on the given set, among which we consider the following:

- $(a, b) \wedge (c, d) := (a \cap c, b \cup d)$,
- $(a, b) \vee (c, d) := (a \cup c, b \cap d)$,
- $(a, b) \cdot (c, d) := (a \cap c, (b \cap d) \cup (a \cap c)')$,
- $\sim (a, b) := (b, a)$,
- $\bot := (0, 1)$,
- $\top := (1, 0)$,
- $e := (1, 1)$.

If we take them as basic operations, it is possible to define a class of algebras of type (2,2,2,1,0,0,0).

In our talk, using well-known results from the literature (see, for example, [5] and [7]), we manage to describe these algebras as members of the subvariety of bounded odd Sugihara monoids generated by the three-element chain, that is, the variety of bounded three-valued Sugihara monoids. Furthermore, any bounded three-valued Sugihara monoid is isomorphic (as a bounded Sugihara monoid) to an assume-guarantee contract algebra. That is to say, the class of assume-guarantee contract algebras generates the variety of bounded three-valued Sugihara monoids.

As a consequence of the aforementioned facts, we get the following result, where BA and $\mathsf{B3SM}$ are the categories of Boolean algebras and bounded three-valued Sugihara monoids, respectively.

Proposition: *Functors* $(\ )^- : \mathsf{B3SM} \rightleftarrows \mathsf{BA} : S$ *witness a categorical equivalence, where* S *is the functor induced by the construction defined above and* $(\ )^-$ *is the functor induced by taking the subalgebra formed by the elements below the identity* e.

Furthermore, we consider other well-studied varieties that are term equivalent to the mentioned variety of bounded three-valued Sugihara monoids and hence, that provide alternative abstract characterizations of assume-guarantee contract algebras in alternative signatures. More concretely, we show that assume-guarantee contract algebras may be regarded either as elements of the variety of centred three-valued double p-algebras (see [2]) or as elements of the variety of centred three-valued Lukasiewicz algebras (see [6]).

In [4], the author finds an adjunction between the category of Boolean algebras and the category ASA of Stone algebras (Heyting algebras satisfying the equation $\neg x \vee \neg\neg x = 1$) expanded with a constant e satisfying the identity $e \to x = \neg\neg x$. He takes the set $C(B)$ of contracts on a Boolean algebra $\mathbf{B}$ as an algebra in ASA with $e = (1, 1)$, not in $\mathsf{B3SM}$. The assignment $\mathbf{B} \mapsto \mathbf{C}(\mathbf{B})$ defines a functor $\mathbf{C} : \mathsf{BA} \to \mathsf{ASA}$. It is shown that $\mathbf{C}$ is part of an adjoint pair $\mathbf{C} \dashv \mathbf{Clos}$, where $\mathbf{Clos}(\mathbf{A})$ is the Stonean subalgebra of an algebra $\mathbf{A}$ in ASA formed by its complemented elements. In any pseudo-complemented bounded distributive lattice $(A; \wedge, \vee, \neg, 0, 1)$ having e as minimum dense element, the sublattice $[e) = \{a \in A : a \leq e\}$, together with the unary operation N defined by $Na := \neg a \wedge e$, is a Boolean lattice isomorphic to $\mathbf{Clos}(A)$. Instead of $\mathbf{Clos}$, the author could have taken the functor $(\ )^+ : \mathsf{ASA} \to \mathsf{BA}$ defined by the assignment $A \mapsto [e)$. Clearly, we also have that $\mathbf{C} \dashv (\ )^+$.

Due to the functional completeness of bounded three-valued Sugihara monoids, it follows that, given a Boolean algebra $\mathbf{B}$, the algebra $\mathbf{S}(\mathbf{B})$ has the underlying structure of a Heyting algebra, which is Stonean and has e as minimum dense element. As a consequence, we have a forgetful functor $\mathbf{U} : \mathsf{B3SM} \to \mathsf{ASA}$ making the following diagram commute.

$$\begin{tikzcd}
\mathsf{BA} \arrow[rr, shift left, "\mathcal{C}"] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left, "\mathbf{S}"] & & \mathsf{ASA} \arrow[ll, shift left, "(\ )^+"] \\
& \mathsf{B3SM} \arrow[ul, shift left, "(\ )^-"] \arrow[ur, "U"'] &
\end{tikzcd}$$

Since $\mathbf{S}$ and $(\ )^-$ witness an equivalence, it follows that $(\ )^- \dashv \mathbf{S}$ and, in consequence, $\mathbf{U} = \mathbf{C} \circ (\ )^- \dashv \mathbf{S} \circ (\ )^+ \cong \mathbf{S} \circ \mathbf{Clos}$. This establishes a relation between our results and those in [4].

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Characterizing Formulas using Post's Lattice

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Given two sets of variable assignments, E^+ and E^- say, over a finite set of propositional variables, along with a propositional formula φ, we assert that φ *fits* E^+, E^- if,

- for every $t \in E^+$, $t \models \varphi$, and
- for every $t \in E^-$, $t \not\models \varphi$.

Moreover, E^+, E^- *uniquely characterizes* φ if, φ fits E^+, E^-, and for every ψ fitting E^+, E^-, ψ is equivalent to φ. It can established, somewhat easily, that for every propositional formula φ, there is a pair E^+_φ, E^-_φ that uniquely characterizes it.

Every truth table over a finite number of propositional variables can be divided into two sets of variable assignments, E^+ and E^- say, representing the *true* and *false* truth assignments of the table, respectively. It follows from a well-known result [3] that there is exactly one formula fitting E^+, E^-, modulo equivalence.

Building upon this result, by fixing a set of propositional variables, PROP say, one can derive an unique characterization of every φ from its truth table (provided that the variables occurring in φ are in PROP). The unique characterization thus obtained should have *all* the variable assignments over the previously fixed set of propositional variables, i.e. PROP[1]. The purpose of this paper, in an informal manner, is to address the question: *What happens to the size of the unique characterization if we consider formulas, not from the full propositional fragment, but within some reduced fragment of propositional logic?*

A *Boolean connective* is function $f : \{0,1\}^n \to \{0,1\}$, where $n \geq 0$. Upon fixing a set O of connectives and a finite set of variables PROP, $\mathrm{PL}_O[\mathrm{PROP}]$ is defined as the smallest class that

- contains all the projections, $\pi^n_k(x_1, \ldots, x_n) = x_k$ for $n \geq k > 0$ and $x_1 \ldots x_n \in \mathrm{PROP}$.
- is closed under composition, $f(x_1, \ldots, x_n), g_1, \ldots, g_n \in \mathrm{PL}_O[\mathrm{PROP}]$ then $f(g_1, \ldots g_n) \in \mathrm{PL}_O[\mathrm{PROP}]$, where $x_1 \ldots x_n \in \mathrm{PROP}$.

The study of fragments then corresponds to study of such aforementioned sets. One good example is that of $\mathrm{PL}_\wedge[\mathrm{PROP}]$[1].

$\mathrm{PL}_\wedge[\mathrm{PROP}]$ doesn't have the property that corresponding to *every* truth table has a fitting formula. But every formula ψ in $\mathrm{PL}_\wedge[\mathrm{PROP}]$ can be uniquely characterized by pair E^+, E^- s.t. $|E^+| + |E^-| \leq |\mathrm{PROP}|$. So indeed there are fragments with better bounds for size of unique characterization.

The preceding example motivates us to play with the bounds for unique characterizations with respect to different fragments. We consider and classify three cases in this paper:

1. The bound on the unique characterization is a bi-variate polynomial in $|\mathrm{PROP}|$ and the size of the formula.
2. The bound on the unique characterization is exponential, but only in the size of the formula.
3. The bound on the unique characterization is a polynomial only in the size of the formula.

The statement of our results requires a little bit of familiarity with Post's Lattice and definition of *clones generated by a set of Boolean connectives*, denoted by $CL(O)$. The statement of our classifications are as follows:

Theorem 0.1. *For any set of Boolean connectives O, the following the equivalent:*

- *There exists a polynomial $p(x,y)$ s.t. for every PROP, and every $\varphi \in PL_O[PROP]$, there is a pair E^+, E^- that uniquely characterizes φ with $|E^+| + |E^-| < p(|\varphi|, |PROP|)$*
- *$CL(O)$ is a subset of either of the three (i) $CL(\wedge, \bot, \top)$, (ii) $CL(\vee, \bot, \top)$ or (iii) $CL(\oplus, \top)$.*

Theorem 0.2. *For any set of Boolean connectives O, the following the equivalent:*

- *For every PROP, and every $\varphi \in PL_O[PROP]$, there is a pair E^+, E^- that uniquely characterizes φ with $|E^+| + |E^-| < 2^{(|\varphi|)}$*
- *$CL(O)$ is a subset of either of the three (i) $CL(\wedge, \bot, \top)$, (ii) $CL(\vee, \bot, \top)$ or (iii) $CL(\oplus, \top)$.*

Theorem 0.3. *For any set of Boolean connectives O, the following the equivalent:*

- *There exists a polynomial $p(x,y)$ s.t. for every PROP, and every $\varphi \in PL_O[PROP]$, there is a pair E^+, E^- that uniquely characterizes φ with $|E^+| + |E^-| < p(|\varphi|)$*
- *$CL(O)$ is a subset of either of the three (i) $CL(\wedge, \bot, \top)$, (ii) $CL(\vee, \bot, \top)$ or (iii) $CL(\oplus, \top)$.*

Although ($\Leftarrow$) direction of the above mentioned results can be established through combinatorial methods, the ($\Rightarrow$) direction requires some sophisticated machinery. We use a special kind of reduction, inspired from [2]. In fact theorem 1.1 has strong correspondence to the main result in [2]. We can refine theorem 1.1 even further based on the techniques used.

Corollary 0.3.1. *For any set of Boolean connectives O, the following the equivalent:*

- *For every PROP, and every $\varphi \in PL_O[PROP]$, there is a pair E^+, E^- that uniquely characterizes φ with $|E^+| + |E^-| < |PROP| + 1$*
- *$CL(O)$ is a subset of either of the three (i) $CL(\wedge, \bot, \top)$, (ii) $CL(\vee, \bot, \top)$ or (iii) $CL(\oplus, \top)$.*

The results we have provided so far are concerned with upper bounds, to finish off we would establish a result on the lower bounds as well. As it turns out, the problem with coming up *reasonable* lower bounds is harder, but we have the following result:

Theorem 0.4. *Any unique characterization E^+, E^- of φ, where $\varphi \in PL_{\oplus}[PROP]$, we get that $|E^+| + |E^-| = |PROP|$.*

Currently we are aiming to extend the results to modal fragments as well, but instead, we are looking at finite characterizations.

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Finitely Generated Varieties of Commutative BCK-algebras: Covers

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BCK-algebras were first introduced in [3] as an algebraic semantics for non-classical logic that uses only implication. Every BCK-algebra admits an ordering, and if it satisfies commutativity law (which in BCK-algebras is not the same as the standard commutativity of binary operation), then the underlying poset is a meet semi-lattice. For the sake of brevity, we will refer to commutative BCK-algebras as "cBCK-algebras". Unlike BCK-algebras, cBCK-algebras form a variety.

The variety of all cBCK-algebras has several noteworthy properties, including congruence distributivity and 3-permutability. In contrast, no nontrivial subvariety is 2-permutable. That is important since having arithmetical variety would facilitate the investigation. Finitely generated varieties of cBCK-algebras are semisimple, i.e. any subdirectly irreducible member is simple. Also, every finite simple cBCK-algebra is hereditary simple. A crucial fact is that subdirectly irreducible cBCK-algebras are (regarding their order) rooted trees [5], [2].

We are interested in covers of finitely generated varieties of cBCK-algebras. Let $\mathcal{V}$ be a finitely generated variety of cBCK-algebras. Then, there exist $\mathbf{A}_1, \ldots, \mathbf{A}_n$ finite subdirectly irreducible cBCK-algebras such that $\mathcal{V} = \mathrm{V}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$. From congruence distributivity, it follows that $\mathcal{V} = \mathrm{V}(\mathbf{A}_1) \vee \cdots \vee \mathrm{V}(\mathbf{A}_n)$. Therefore, investigating covers of finitely generated varieties can be reduced to investigating covers of varieties generated by a single finite subdirectly irreducible cBCK-algebra. From now on, let $\mathcal{V} = \mathrm{V}(\mathbf{A})$, where $\mathbf{A}$ is finite simple subdirectly irreducible cBCK-algebra. An important observation is that $\mathrm{Si}(\mathcal{V})$ (subdirectly irreducible members of $\mathcal{V}$) consists (up to isomorphisms) only of $\mathrm{S}(\mathbf{A})$ (subalgebras of $\mathbf{A}$).

The fact that $\mathrm{Si}(\mathcal{V}) = \mathrm{S}(\mathbf{A})$ motivates us to first explore $\mathrm{S}(\mathbf{A})$. There are two kinds of subalgebras: downsets and the others. The others can be characterised as a set of elements of $\mathbf{A}$ that have height divisible by some integer $k > 1$. Under some conditions, such a set indeed forms a subalgebra. The detailed characterisation is the subject of the first part of the talk.

The second part of the presentation focuses on the covers. The goal is to find all covers of $\mathcal{V}$, i.e. to find subdirectly irreducible cBCK-algebras that generate the covers. The construction involves considering all subalgebras of $\mathbf{A}$ and then considering their extensions by adding a leaf to some vertex (not the root). We prove that by the construction, we obtain a cover and that every cover is achievable by the construction.

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The Logic with Unsharp Implication and Negation - Algebraic Approach

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It is well-known that intuitionistic logics can be formalized by means of Brouwerian semilattices, i.e. relatively pseudocomplemented semilattices. Then the logical connective implication is considered to be the relative pseudocomplement and conjunction is the semilattice operation meet. If the Brouwerian semilattice has a bottom element 0 then the relative pseudocomplement with respect to 0 is called the pseudocomplement and it is considered as the connective negation in this logic. Our idea is to consider an arbitrary meet-semilattice with 0 satisfying only the Ascending Chain Condition, which is trivially satisfied in finite semilattices, and introduce the connective negation x^0 as the set of all maximal elements z satisfying $x \wedge z = 0$ and the connective implication $x \to y$ as the set of all maximal elements z satisfying $x \wedge z \leq y$. The Ascending Chain Condition means that every chain has a maxima element and it ensures that every non-void subset has maximal elements. Such a negation and implication are "unsharp" since they assign respectively, to one entry x or to two entries x and y belonging to the semilattice, a subset instead of an element of the semilattice. Surprisingly, these kind of negation and implication, respectively, still share a number of properties of the corresponding connectives in intuitionistic logic, in particular the derivation rule Modus Ponens. Moreover, unsharp negation and unsharp implication can be characterized by means of five, respectively seven simple axioms. Several examples are presented. The concepts of a deductive system and of a filter are introduced as well as the congruence determined by such a filter. We finally describe certain relationships between these concepts.

AMS Subject Classification: 03G10, 03G25, 03B60, 06A12, 06D20

Keywords: Semilattice, Brouwerian semilattice, Heyting algebra, intuitionistic logic, unsharp negation, unsharp implication, deductive system, filter, congruence

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Pretabular Tense Logics over $\mathsf{S4}_t$

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A logic L is called tabular if $L = \mathsf{Log}(\mathfrak{A})$ for some finite algebra $\mathfrak{A}$. L is called pretabular if L itself is not tabular while all of its proper consistent extensions are tabular. Let $\mathsf{Pre}(L)$ denote the set of pretabular logics extending L. It is proved in [7] that $|\mathsf{Pre}(\mathsf{Int})| = 3$. It was shown in [8, 4] that $|\mathsf{Pre}(\mathsf{S4})| = 5$. Moreover, [1] proved that $|\mathsf{Pre}(\mathsf{K4})| = 2^{\aleph_0}$. However, the tense case is more involved and we know much less about it. [6] introduced a pretabular tense logic $\mathsf{Ga} \in \mathsf{NExt}(\mathsf{S4}_t)$, whose frames have a maximum depth and width of 2 and do not contain any proper clusters.[1] It is claimed in [10] that $|\mathsf{Pre}(\mathsf{S4}_t)| \geq \aleph_0$ without a proof.

In this work, we study pretabular tense logics in the lattice $\mathsf{NExt}(\mathsf{S4}_t)$. We start with the sublattice $\mathsf{NExt}(\mathsf{S4.3}_t)$, where $\mathsf{S4.3}_t = \mathsf{S4} \oplus \{\boxtimes(\boxtimes p \to q) \vee \boxtimes(\boxtimes q \to p) : \boxtimes \in \{\Box, \blacksquare\}\}$ is the tense logic of chains. It turns out that the lattice $\mathsf{NExt}(\mathsf{S4.3}_t)$ is already much more complex than the lattice $\mathsf{NExt}(\mathsf{S4.3})$. It was shown in [5, 2] that every modal logic in $\mathsf{NExt}(\mathsf{S4.3})$ is finitely axiomatizable and enjoys the finite model property. However, $\mathsf{NExt}(\mathsf{S4.3}_t)$ contains infinitely many incomplete tense logics (see [11]). We obtain a full characterization of pretabular tense logics over $\mathsf{S4.3}_t$ as follows:

Theorem 1. *There are exactly five pretabular tense logics in* $\mathsf{NExt}(\mathsf{S4.3}_t)$*. More precisely,*

$$\mathsf{Pre}(\mathsf{S4.3}_t) = \{L_i : i < 5\}, \text{ where } L_i = \textstyle\bigcap_{n\in\omega} \mathsf{Log}_t(\mathfrak{C}_i^n).^{2}$$

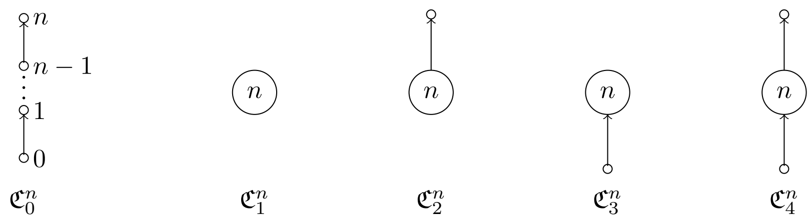


Figure 1: Frames $\mathfrak{C}_i^n$

It is clear that $\mathsf{Pre}(\mathsf{S4.3}) = \{\bigcap_{n\in\omega} \mathsf{Log}_\Diamond(\mathfrak{C}_i^n) : i < 3\}$, where $\mathsf{Log}_\Diamond(\mathfrak{C}_i^n)$ is the modal logic of $\mathfrak{C}_i^n$. The interaction between tense operators lead to new pretabular logics L_3 and L_4.

We generalize the results above and consider the lattices $\mathsf{NExt}(\mathsf{S4.3}_t^+)$ and $\mathsf{NExt}(\mathsf{S4.3}_t^-)$, where $\mathsf{S4.3}_t^+ = \mathsf{S4} \oplus \Box(\Box p \to q) \vee \Box(\Box q \to p)$ and $\mathsf{S4.3}_t^- = \mathsf{S4} \oplus \blacksquare(\blacksquare p \to q) \vee \blacksquare(\blacksquare q \to p)$. The bi-intuitionistic logic of 'co-trees' was studied in [9]. $\mathsf{S4.3}_t^+$ and $\mathsf{S4.3}_t^-$ are the tense logics of 'co-trees' and 'trees', respectively. The main result we have for them is as follows:

Theorem 2. $|\mathsf{Pre}(\mathsf{S4.3}_t^+)| = |\mathsf{Pre}(\mathsf{S4.3}_t^-)| = 12$.

Pretabular tense logics in $\mathsf{Pre}(\mathsf{S4.3}_t^-) \setminus \mathsf{Pre}(\mathsf{S4.3}_t)$ are characterized by the classes of finite frames given in Figure 2. In the modal case, only the forks generate a pretabular logic.

[1] $\mathsf{S4}_t$ is the tense logic of reflexive and transitive frames.

[2] $\mathfrak{C}_i^n$ are frames depicted in Figure 1. n in the figures denotes a cluster with n points.

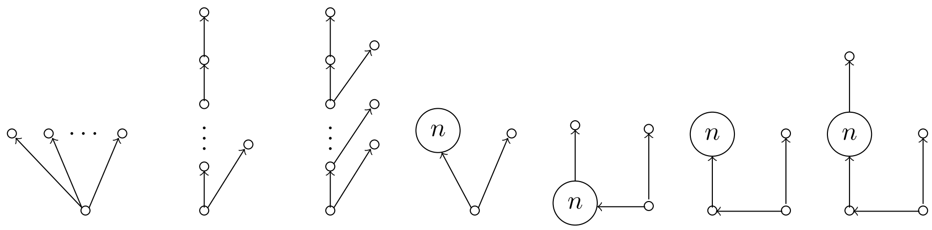
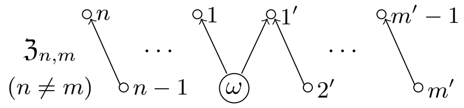


Figure 2: Frames for logics in $\mathsf{Pre}(\mathsf{S4.3}_t^-) \setminus \mathsf{Pre}(\mathsf{S4.3}_t)$

Once we allow 'zigzag-like' frames, we are in completely different situation, even if we put strong constraints on the depth and width of the frames. We consider the tense logic $\mathsf{BS}^2_{2,2} = \mathsf{S4}_t \oplus \{\mathsf{bd}_2, \mathsf{bw}_2^+, \mathsf{bw}_2^-\}$, where $\mathsf{bd}_2, \mathsf{bw}_2^+$ and bw_2^- are defined as in [3]. $\mathsf{BS}^2_{2,2}$ is exactly the tense logic of 'zigzags' with clusters. We obtain also a full characterization of pretabular tense logics in $\mathsf{NExt}(\mathsf{BS}^2_{2,2})$ as follows:

Theorem 3. *Let $L \in \mathsf{NExt}(\mathsf{BS}^2_{2,2})$. Then L is pretabular if and only if $L = \mathsf{Ga}$ or $L = \mathsf{Log}(\mathfrak{F})$ for some $\mathfrak{F} \in \mathcal{Z} \cup \breve{\mathcal{Z}}$, where $\mathcal{Z}$ is the class of frames depicted in the figure below.*[3]



Corollary 4. $|\mathsf{Pre}(\mathsf{BS}^2_{2,2})| = \aleph_0$.

We construct infinitely many pretabular tense logics in $\mathsf{NExt}(\mathsf{S4}_t)$, which provides a proof for the claim in [10]. The next step is to investigate the set $\mathsf{Pre}(\mathsf{S4}_t)$ of pretabular logics and our conjecture is that $|\mathsf{Pre}(\mathsf{S4}_t)| = 2^{\aleph_0}$.

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[3] $\breve{\mathcal{Z}} = \{(W, R^{-1}) : (W, R) \in \mathcal{Z}\}$ and ω in the figure denotes a cluster with ω points.

A Categorical Representation of Thin Trees

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Abstract

Infinite trees with countably many branches, called thin trees, have been studied via methods from automata theory and algebra. We take a categorical approach to thin trees using the framework of coalgebra. We show that the collection of thin trees can be seen as an initial algebra satisfying a certain axiom. We prove this by defining an algebra of thin tree representatives and showing that each thin tree has a canonical representative.

1 Background

Infinite words serve as a basis for the analysis of reactive systems and have been studied by means of automata and algebraic recognition [6]. Subsequently, infinite trees have also become an object of study, as they can express more complex systems where branching occurs. Tools from automata theory have been adapted to infinite trees [4].

In [7, 5] the authors look into automata and algebras for a class of infinite trees called *thin trees*. These are the trees that have countably many infinite branches. Every node in a thin tree can be assigned an ordinal called *rank*, which allows for inductive reasoning on the rank of thin trees. Moreover, languages of thin trees admit an algebraic characterisation via *thin algebras*, which are an extension of the notion of an ω-semigroup for infinite words. Thin algebras and induction on the rank are used to prove that languages of thin trees are recognised by *unambiguous automata*, i.e., automata that have unique accepting runs.

2 Contribution

In our current work we employ category theory to provide a uniform account of thin trees for a finite signature F. We base our approach on the formalisation of trees via F-*coalgebras* for a polynomial functor F over Set (see, e.g., [3]). Indeed, every tree with branching type F corresponds to an element of the final F-coalgebra (Z, ζ), and every element of (Z, ζ) can be unravelled into a tree. We take a look into the subcoalgebra (Z^{Th}, ζ^{Th}) of (Z, ζ) consisting of those elements whose unravelling is a thin tree. By endowing Z^{Th} with a suitable algebraic structure β^{Th}, we characterise (Z^{Th}, β^{Th}) as the initial object in a certain category of algebras ThAlg. In this way, we capture the principle of induction on the rank of thin trees via the universal property of initiality. Moreover, objects in ThAlg allow for algebraic recognition of languages of thin trees, analogously to thin algebras in [7]. This paves the way for future work on categorifying properties of thin trees, such as the existence of unambiguous automata.

3 Universal Property of Thin Trees

Here we give some details behind the construction of (Z^{Th}, β^{Th}). A key ingredient is the *functor derivative* F' [1], which represents the type of *contexts*, i.e., tree nodes where one successor is replaced by a hole. A context c can be composed with a tree or another context c' by plugging c' into the hole of c.

We define the type of streams of F-contexts $G \coloneqq (F')^\omega$ and denote the initial $(F+G)$-algebra by (A, α). Every term in A can be seen as a representative of a tree. If a term has type F, we interpret it as a tree node with given immediate successors. If it has type G, we interpret it as the tree obtained by composing infinitely many F-contexts. This gives rise to an *interpretation map* $\mathit{int} : A \to Z$. Moreover, we observe that interpretations are thin, i.e., $\mathit{int}[A] \subseteq Z^{Th}$. However, one element of Z^{Th} can have many representatives in A. For example, consider an element of Z^{Th} whose unravelling consists of a single infinite branch. It can be represented as a stream x_1 of contexts whose only successor is the hole, or as a node x_2 whose only successor is the stream x_1.

In order to get unique representatives, we quotient (A, α) by the congruence $\approx$ generated by the following axiom (†). We identify a term x of type G with the term y of type F obtained by plugging $\mathit{tail}(x)$ into the context $\mathit{head}(x)$. For instance, (†) will directly identify x_1 and x_2 from the example above. We show that $\approx$ is sound for int, i.e., terms identified by $\approx$ have the same interpretation. In order to show that each element of Z^{Th} is represented by a unique equivalence class of $\approx$, we introduce the notion of a *normal term*. It is defined via the *rank* of a term, which is the earliest step in the initial colimit construction of (A, α) at which the term appears. Now a term is called normal if it has the least rank among the terms with the same interpretation, and all its subterms are normal.

We prove two main results: (1) each element of Z^{Th} has a unique normal representative, and (2) the quotient relation $\approx$ identifies each term with its corresponding normal term. As a result, we conclude that the quotient of (A, α) by $\approx$ contains a unique representative for each element of Z^{Th}. Thus, for a suitable $(F+G)$-algebra structure β^{Th}, we have that (Z^{Th}, β^{Th}) is isomorphic to the quotient of (A, α) by $\approx$, so (Z^{Th}, β^{Th}) is initial among all $(F+G)$-algebras satisfying the axiom (†).

Beyond the purposes of our proofs, we hope that thin tree representatives can find use in applications, such as automata learning, where algorithms are sensitive to the particular presentation of objects [2].

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Logics for Probabilistic Dynamical Systems

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Our main objective is to investigate (discrete-time) Markov stochastic processes augmented by a dynamic mapping from the modal logic point of view. These mathematical structures, which we call *dynamic Markov processes*, are of the form $\langle \Omega, \mathcal{A}, T, f\rangle$ where $\langle \Omega, \mathcal{A}\rangle$ is a measure space, $T : \Omega \times \mathcal{A} \to [0, 1]$ is a Markov kernel and $f : \Omega \to \Omega$ is a measurable function. In this case, the triple $\langle \Omega, \mathcal{A}, T\rangle$ is called a *Markov process* on the state space $\langle \Omega, \mathcal{A}\rangle$.

In a somewhat broader context, the notion of *probabilistic (random) dynamical systems* [1] covers one of the most important classes of dynamical systems with probabilistic features. Typically, these systems contain stochastic processes, e.g. Markov processes possibly augmented by some additional dynamic structures that describe the dynamic behavior of the system. In a sense, our investigations lay in logical descriptions of certain special cases of probabilistic dynamical systems. These structures have diverse applications, from stochastic differential equations to finance and economics [4].

There are various logical approaches to modeling probability structures, among which we consider propositional modal logic. In this approach, bounds on probability are treated as modal operators. So there are countably many probability modal operators L_r, for each $r \in \mathcal{Q} \cap [0, 1]$. For a formula φ, the formula $L_r\varphi$ is interpreted as 'the probability of φ is at least r'. The resulting *modal probability logic* is denoted by PL. It is shown that this logic is decidable [7, 13]. There are numerous papers in this area dealing with axiomatization which demonstrate several completeness for PL [2, 7, 8, 10, 13] and prove some nice semantical properties [6, 12]. There is also infinitary version of PL denoted by PL_{ω_1} [3, 9, 11]. The language PL_{ω_1} extends the language PL by adding (infinite) countable conjunctions and disjunctions.

This presentation, which is based on our recent work in [5], is divided into two parts. The first part of our research is devoted to introducing the finitary *dynamic probability logic* (DPL). The language of DPL is obtained by adding a temporal-like modal operator $\bigcirc$ (denoted as dynamic operator) which describes the dynamic part of the system. We subsequently propose a Hilbert-style axiomatization for this logic and demonstrate its strong completeness for the class of all dynamic Markov processes based on standard Borel spaces[1]. To this end, we use a canonical model construction based on special maximal finitely consistent subsets of formula called *saturated sets*. This approach is inspired by the proof of strong completeness for Markovian logics in [10]. We further examine the logics of some important subclasses of dynamic Markov processes, including the class of all dynamic Markov processes of the form $\langle \Omega, \mathcal{A}, T, f\rangle$ that are *measure-preserving*, i.e., $T(w, f^{-1}(A)) = T(f(w), A)$ for each $w \in \Omega$ and $a \in \mathcal{A}$. We also present a logic for the class of all *abstract dynamical systems*, i.e. structures of the form $\langle \Omega, \mathcal{A}, \mu, f\rangle$ where $\langle \Omega, \mathcal{A}, \mu\rangle$ is a probability space and $f : \Omega \to \Omega$ is a measure-preserving function.

Our ideas naturally extend to introducing the *infinitary dynamic probability logic*. This logic, which is denoted by DPL_{ω_1}, allows countable conjunctions and disjunctions. The expressive power of DPL_{ω_1} is compatible with σ-additivity of probability measures. So within this logic,

*Speaker

[1] A measure space $\langle \Omega, \mathcal{A}\rangle$ is called a *standard Borel space* if $\mathcal{A}$ is the Borel σ-algebra generated by a Polish topology on Ω.

many properties of probability can be naturally axiomatized, and hence, it is not hard to extend ideas from [3, 9] to show that there exists a weakly complete Hilbert-style axiomatization for this logic. Meanwhile, we show that whenever the logic is restricted to its countable fragments, the proposed axiomatization is strongly complete for the class of all dynamic Markov processes. We should point out that while the canonical model introduced for the proof of strong completeness for each countable fragment A of PL_{ω_1} in [9, Subsection 5.2] depends on A, we show that the canonical model of DPL can be served uniformly as a canonical model for each countable fragment of DPL_{ω_1}.

The second contribution of the present research is allocated to investigating (frame) definability of natural properties of dynamic Markov processes. We show that some dynamic properties such as measure-preserving, ergodicity, and mixing are definable within DPL and DPL_{ω_1}. Moreover, we consider the *infinitary probability logic with initial distribution* (InPL_{ω_1}) by disregarding the dynamic operator. This logic studies *Markov processes with initial distribution*, i.e. structures of the form $\langle \Omega, \mathcal{A}, T, \pi \rangle$ where $\langle \Omega, \mathcal{A}, T \rangle$ is a Markov process and $\pi : \mathcal{A} \to [0,1]$ is a σ-additive probability measure. We show that the strong expressive power of InPL_{ω_1} would allow us to define *n-step transition probabilities* T^n of Markov kernel T. From this, we conclude that many natural stochastic properties of Markov processes such as stationary, invariance, irreducibility, and recurrence can be stated within InPL_{ω_1}. These results particularly show that DPL as well as DPL_{ω_1} are natural and important extensions of PL.

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Gödel–Dummett CTL

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Gödel–Dummett logic is a well-known and extensively studied multivalued logic [5]. It is both a superintuitionistic logic and a t-norm fuzzy logic. Computation tree logic (CTL) [4] is a branching-time temporal logic that is a relative of linear temporal logic (LTL) (both are fragments of CTL*). Both LTL and CTL were designed and have been used very successfully for formal verification.

Although nonclassical variants of modal and temporal logics often compare unfavourably to their classical counterparts in terms of logical and computational properties [6, 3], recent investigations have shown that Gödel–Dummett logic pairs well with linear temporal logic. Indeed the variant of LTL whose modality-free fragment is Gödel–Dummett logic is not only decidable, but has an optimal PSPACE complexity [2], and a finite Hilbert-style calculus has been given for Gödel–Dummett LTL enriched with the "coimplication" connective of bi-intuitionistic logic [1].

In this talk we report on similar investigations into a Gödel–Dummett CTL and show that it too is decidable.

Fix a countably infinite set $\mathbb{P}$ of propositional variables. Then the **bi-intuitionistic CTL language** $\mathcal{L}$ is the language defined by the grammar (in Backus–Naur form):

$$\varphi \coloneqq p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi \prec \varphi \mid \exists \mathsf{X}\varphi \mid \forall \mathsf{X}\varphi \mid \exists \mathsf{G}\varphi \mid \forall \mathsf{F}\varphi \mid \exists(\varphi \mathbin{\mathsf{U}} \varphi) \mid \forall(\varphi \mathbin{\mathsf{R}} \varphi),$$

where $p \in \mathbb{P}$. Here, an $\exists$ is read as 'there exists a path (from this state)', a $\forall$ as 'for all paths', X is as 'ne**x**t', G as '**g**oing (to always be)', F as '**f**uture', U as '**u**ntil' and R as '**r**eleased by'. The connective $\prec$ is *co-implication* and represents the operator that is dual to implication [7]. We can also define the following abbreviations:

- $\top$ abbreviates $p \to p$, and $\bot$ abbreviates $p \prec p$, for some fixed, but unspecified, $p \in \mathbb{P}$;
- $\neg\varphi$ abbreviates $\varphi \to \bot$;
- $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \to \psi) \wedge (\psi \to \varphi)$ (not the formula $(\varphi \to \psi) \wedge (\varphi \prec \psi)$);
- $\forall\mathsf{G}\varphi$ abbreviates $\forall(\varphi \mathbin{\mathsf{R}} \bot)$ and $\exists\mathsf{F}\varphi$ abbreviates $\exists(\top \mathbin{\mathsf{U}} \varphi)$;
- $\forall(\varphi \mathbin{\mathsf{U}} \psi)$ abbreviates $\forall(\varphi \mathbin{\mathsf{R}} \psi) \wedge \forall\mathsf{F}\psi$ and $\exists(\varphi \mathbin{\mathsf{R}} \psi)$ abbreviates $\exists(\varphi \mathbin{\mathsf{U}} \psi) \vee \exists\mathsf{G}\psi$;

We define the Gödel–Dummett CTL logic using two natural semantics (the details of which we do not give here): first a *real-valued semantics*, where statements have a degree of truth in the real unit interval and second a *bi-relational semantics*.

We define:

- the logic $\mathsf{GCTL}_{\mathbb{R}}$ to be the set of $\mathcal{L}$-formulas that are valid with respect to the real-valued semantics;
- the logic $\mathsf{GCTL}_{\mathrm{rel}}$ to be the set of $\mathcal{L}$-formulas that are valid with respect to the bi-relational semantics.

However, any formula falsifiable on a real-valued model is falsifiable on a bi-relational model.

Proposition 1. $\mathsf{GCTL}_{\mathrm{rel}} \subseteq \mathsf{GCTL}_{\mathbb{R}}$.

For $\mathsf{GCTL}_{\mathrm{rel}}$, we use a variant of the technical notion of a pseudo-model, as introduced in [4], and adapted here for CTL. We show that every bi-relationally falsifiable statement is falsifiable on a finite pseudo-model, and vice versa. This directly yields an algorithm for deciding if a statement is valid or not.

Theorem 2. *The logic* $\mathsf{GCTL}_{\mathrm{rel}}$ *is decidable.*

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The Kuratowski's Problem in Pointfree Topology

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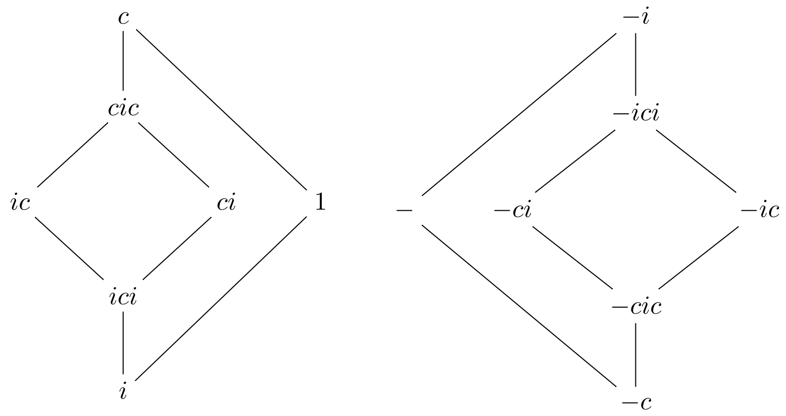
A classic result of Kuratowski states that there are at most 7 distinct combinations of the operators of interior (i) and closure (c) on a topological space, which become 14 if also the set-theoretic complement ($-$) is considered.[1] These operators form an ordered monoid w.r.t. composition and pointwise ordering, the so-called Kuratowski's monoid, whose Hasse diagram is shown below:



(where 1 is the identity operator). Special classes of spaces can be characterized by the fact that two or more of these operators coincide [4]; for instance, a space whose open sets form a complete Boolean algebra satisfies the equation $ici = i$.

> What happens to this picture if it is looked at from a constructive point of view?
> And what about the pointfree (i.e. localic) version of the Kuratowski's problem?

We answer both of these questions and we explain why they are related to each other.

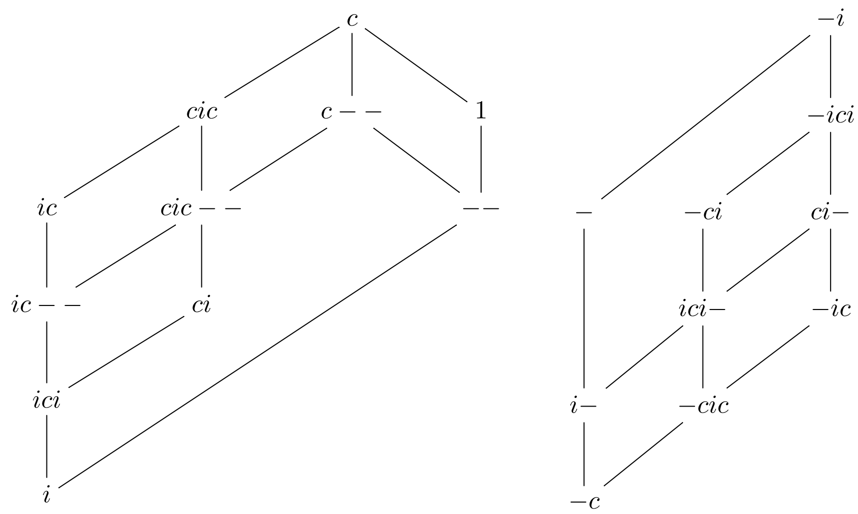
First, we recall a constructive account of the closure-interior problem (that is, the one not involving the set-theoretic complement) that we know from Giovanni Sambin [3]. For the sake of generality, we consider a closure operator and an interior operator on an arbitrary poset. It turns out that the ordered monoid generated by i and c in such a framework depends neither on the Law of Excluded Middle nor on topological notions as strictly understood.

In a constructive setting, the collection of subsets of a given set is only a frame (a.k.a. a complete Heyting algebra), instead of a complete Boolean algebra, and the set-theoretic "complement" is only a pseudocomplement. This naturally poses a general version of the Kuratowski's problem on an arbitrary frame, which has potential applications in constructive modal

[1] In logical terms, this means that there are exactly 13 different modalities (not counting identity) in $S4$.

logic. However, the presence of the pseudocomplement greatly increases the number of possible combinations [1]. To simplify the matter we restrict to the case in which $c = -i-$ (this equation is constructively true in all topological spaces, although its dual $i = -c-$ is not): this we call the interior-pseudocomplement problem on a frame. Contrary to the Boolean case, we get 31 possible combinations (instead of 14) that apparently are all different [2], in general.

This constructive result can be applied to solve the Kuratowski's problem in a pointfree framework, that is, within the theory of locales. Indeed, it is well known that the sublocales of a given locale form a co-frame (the opposite of the frame of nuclei); in particular, every sublocale has a co-pseudocomplement. Moreover, the usual notion of an open (closed) sublocale gives rise to an interior (a closure) operator on the co-frame of sublocales.[2] So the Kuratowki's problem for sublocales is related, although in a dual way, to the constructive interior-pseudocomplement problem discussed above. We can therefore apply the previous result and, thanks to some specific properties of open/closed sublocales, we can lower the number of possible combinations of interior, closure and co-pseudocomplement to 21 (see the picture below).



Showing that this picture cannot be further simplified is still an open problem [2].

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From higher-order rewriting systems to higher-order categorial algebras and higher-order Curry-Howard isomorphisms

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In proof theory and programming language theory, the Curry-Howard correspondence explains the direct relationship between computer programs and mathematical proofs. Curry, in [3], was the first to recognize the analogy between combinatory logic and the axioms of a Hilbert-style deductive system for positive propositional logic. Later, Howard [4] observed a similar formal analogy between the lambda calculus and the proof rules of a Gentzen-style natural deduction system for propositional logic. The Curry-Howard correspondence associates each proof in intuitionistic logic with a term in Curry's combinatory logic or Church's lambda calculus. This correspondence, also called *proofs-as-programs*, connects intuitionistic logic proofs with terms in combinatory logic or lambda calculus. Essentially, it reveals that deduction systems and computation models are fundamentally the same mathematical entities. The Curry-Howard correspondence sparked research leading to dual-purpose formal systems—serving as both proof systems and typed functional programming languages. Examples include Martin-Löf's intuitionistic type theory [5] and Coquand's Calculus of Constructions [2]. These systems treat proofs as regular objects, allowing properties to be declared about proofs, akin to any other program—an area known as modern type theory. In particular, the Homotopy Type Theory (HoTT) [6] is a new field of mathematical study that combines various aspects of type theory and homotopy theory, incorporating ideas from algebraic topology and homological algebra for the examination of formal derivation systems.

The work to be presented here is based on an ongoing study [1], which will serve as the foundation. In this work, we consider a signature Σ, a set of variables X, and a fixed set $\mathcal{A}$ of rewriting rules. This set $\mathcal{A}$ consists of pairs $\mathfrak{p} = (M, N)$, where M and N are terms in the free Σ-algebra $\mathrm{T}_{\Sigma}(X)$. These pairs are used to form a set of admissible rewriting rules in a formal derivation system $\boldsymbol{\mathcal{A}} = (\Sigma, X, \mathcal{A})$, which leads to the concept of a path between terms. A path is a finite sequence of terms in which, at each step, a rewriting rule is applied. In other words, the $(i+1)$-th term is obtained by substituting subterm M in the i-th term with term N. This object is understood as a simplified version of a proof where, at each step, a derivation rule admitted in the system is used. This leads to a category of paths, where the objects are terms in $\mathrm{T}_{\Sigma}(X)$, and the morphisms are paths between terms. Furthermore, it is shown that the set of paths for a rewriting system $\boldsymbol{\mathcal{A}}$ has the structure of a Σ-algebra and is equipped with an artinian order that specifies the complexity of the path and aids in the inductive study of these objects.

Next, we consider $\Sigma^{\boldsymbol{\mathcal{A}}}$, an extension of the original signature Σ, which includes both categorical operations and rewriting rules from $\mathcal{A}$. For this extension, it is proven that the set of paths has the structure of a partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra. With the assistance of the artinian order on paths, each path is associated with a term in the free $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$, akin to the Curry-Howard construction. This term captures the syntactic derivation that occurs in each path, except for the ordering in the derivation that occurs in parallel. The study of the kernel of the Curry-Howard application reveals that it is a closed $\Sigma^{\boldsymbol{\mathcal{A}}}$-congruence, which allows for the algebraic study of the respective quotient of paths. It has been shown that the quotient of paths by the kernel of the Curry-Howard application has the structure of a category, a partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra, and a partially ordered set with an artinian partial order. This allows for the preservation of the inductive study of path classes. Moreover, there is a strong relationship between the categorical and algebraic structures, as the operations from the original signature Σ act as functors on the categorical structure. The fundamental result establishes that this quotient structure is the free partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra for a variety $\boldsymbol{\mathcal{V}}$ of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebras, subject to equations related to both the categorical and algebraic structures, that is, a Curry-Howard isomorphism type result.

In the second part of this work, we generalize the previous results by considering second-order rewriting systems $\boldsymbol{\mathcal{A}}^{(2)} = (\boldsymbol{\mathcal{A}}^{(1)}, \mathcal{A}^{(2)})$, where $\boldsymbol{\mathcal{A}}^{(1)}$ is a rewriting system and $\mathcal{A}^{(2)}$ is a set of second-order rewriting rules. This helps us introducing the notion of second-order paths, in analogy to homotopies in topological spaces. Analogous results to those in the first part are established, thanks to the definition of a second-order Curry-Howard mapping. Particularly, for a second-order categorical signature $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$, a quotient set with the structure of a 2-category, a partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra, and a partially ordered set with an artinian partial preorder has been constructed. The fundamental result establishes that this quotient structure is the free partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra for a variety $\boldsymbol{\mathcal{V}}^{(2)}$ of second-order partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebras, subject to equations related to the 2-categorical and algebraic structure. In other words, we obtain a second-order Curry-Howard isomorphism for second-order rewriting systems.

This will ultimately lead, in future versions of this work, to the development of a theory aimed at investigating the relationship between ω-rewriting systems and ω-categorial algebras through higher-order Curry-Howard isomorphisms.

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The logic of vague categories

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Categories are cognitive tools humans use to make sense of the world, and interact with it and with each other. They are key to the development and use of language, the construction of knowledge and identity, the formation of evaluation, and decision-making. The literature on categorization is expanding rapidly in fields ranging from cognitive linguistics to social science, management science and AI.

A key issue to the development of the foundations of categorization theory concerns the formalization of the *vague nature* of categories. While mathematical concepts such as 'prime number' or 'circle' have a precise extension, this is not so for concepts such as 'red', 'tall', 'heap' or 'house'. *Vague categories* and concepts admit borderline cases, namely cases for which it is not clear whether the concept should apply or not. For instance, where is the limit between dark blue and light blue? Is a certain object blue, or is it grey or green? The absence of clear-cut boundaries between categories is the main reason why, in most real-life categorization processes, objects are assigned to more than one category, giving rise to the phenomenon of *category-spanning*, which has important consequences on decision-making.

Rough set theory [16] provides the starting point of the formal approach to vagueness proposed in the present contribution, since it accounts for the absence of clear-cut categorical boundaries via the interval induced by the *upper* and *lower approximations* of sets and predicates, arising from an indiscernibility relation on a domain of discourse. In [10], these insights have been extended to the formal environment of *conceptual approximation spaces*, a common generalization of Pawlak's approximation spaces and Wille's formal contexts (aka polarities) [14], on which the present contribution directly builds.

Specifically, the present contribution continues a line of research aimed at introducing and studying logical frameworks specifically designed to reason about categories and categorization, and at using these logics to formalize notions and analyze problems involving categorization arising across disciplines. In [7], building on the general mathematical framework for non-distributive logics developed in [12, 11], the basic normal non-distributive modal logic and some of its axiomatic extensions are interpreted as *epistemic logics of categories and concepts*, and in [8], the corresponding 'common knowledge'-type construction is used to give an epistemic-logical formalization of the notion of *prototype* of a category; in [10, 15], conceptual approximation spaces are proposed as a relational semantics for non-distributive modal logic, which, being interpreted in this context as the logic of *rough concepts*, serves as an encompassing framework for the integration of Rough Set Theory [16] and Formal Concept Analysis (FCA) [14]. Other different but closely related semantics for non-distributive modal logic have been introduced and explored in [4, 6], and generalized to the many-valued semantic setting [5, 13].

In this contribution, building on Bělohlávek's framework of fuzzy formal concepts [1, 2], we present the mathematical and conceptual investigation of the *many-valued polarity-based* relational semantics for non-distributive modal logic. This framework has been initially investigated in [10, Section 7.2]. Further developments in the direction of correspondence theory have been developed in [9], and in [3] it has been applied in the development of unsupervised learning algorithms for outlier detection that also provide explanations of their results.

In our presentation we will discuss the many-valued non-distributive modal logics described above. We will introduce many-valued enriched formal contexts; introduce the semantics and proof theory for the logics; expand on the completeness of this logic; and present results in correspondence and duality in this context. Finally, we will present a generalization of this framework to a framework where the algebra of values is a non-commutative quantale. We will discuss how this shift affects the aforementioned notions and present some further results on correspondence and completeness.

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A 2-categorical analysis of context comprehension

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The problem of modelling the structural rules of type dependency using categories has motivated the study of several structures, varying in generality, occurrence in nature, and adherence to the syntax of dependent type theory. One aspect, that involving free variables and substitution, is neatly dealt with using (possibly refinements of) Grothendieck fibrations. The other main aspect of type dependency is the possibility of making assumptions as encoded in the two rules below

$$\frac{\Gamma \vdash A\ \mathtt{Type}}{\vdash \Gamma.A\ \mathtt{ctx}} \qquad\qquad \frac{\Gamma \vdash A\ \mathtt{Type}}{\Gamma.A \vdash \mathtt{v}_A : A}$$

where the first one (*context extension*) extends the context Γ with the type A, and the second one (*assumption*) provides a "generic term" of A in context $\Gamma.A$. In the first order setting, they allow us to add assumptions to a context, and to prove what has been assumed, respectively.

We present a purely 2-categorical comparison of the two main categorical accounts of these two rules: Jacobs' comprehension categories [Jac99] and Dybjer's categories with families [Dyb96]. They differ in that the former gives prominence to context extension, and the latter to assumption. The comparison itself consists of a biequivalence of 2-categories, which generalises the classical 1-equivalence between the discrete versions of these structures due to Hofmann [Hof97].

The biequivalence goes via a third 2-category of a less known structure called *weakening and contraction comonad*. These appear already in [Jac99, Definition 9.3.1], where Jacobs uses them to justify the definition of comprehension category [Jac99, Theorem 9.3.4]. We call them *w-comonads* for short. On the other hand, categories with families can be formulated as a pair of discrete fibrations over the same base connected by a (suitable) adjunction. This is known thanks to the observations (and proofs) of, among others, Fiore [Fio08], Awodey [Awo18], and Uemura [Uem23, Section 3]. In order to have a uniform comparison with comprehension categories, we drop the assumption of discreteness on the two fibrations and call the resulting structure a *generalised category with families*.

Morphisms of these structures can vary according to the degree of preservation of the relevant structure. We use the well-established taxonomy of morphisms of adjunctions and of (co)monads [KS74, Str72] to classify morphisms of comprehension categories and of generalised categories with families according to the degree of preservation of context comprehension. In particular, this classification entails that there is a single notion of morphism of which all those that have appeared in the literature are particular cases.

Categories with families are in bijection with discrete comprehension categories because, for every object A of $\mathcal{U}$, the objects of $\dot{\mathcal{U}}$ mapped to A (the terms) are in bijection with

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Figure 1: The underlying diagrams in **Cat** of, from left to right, a comprehension category, a w-comonad, and a generalised category with families.

$$
\begin{tikzcd}
\mathcal{U} \arrow[rr, "\chi"] \arrow[dr, "u"'] & & \mathcal{B}^2 \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{B} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{U} \arrow[loop left, "K"] \arrow[d, "u"] \\
\mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\dot{\mathcal{U}} \arrow[rr, bend right, "\Sigma"'] \arrow[rr, phantom, "\top"] \arrow[dr, "\dot{u}"'] & & \mathcal{U} \arrow[ll, bend right, "\Delta"'] \arrow[dl, "u"] \\
& \mathcal{B} &
\end{tikzcd}
$$

sections of the display map χA. In general, sections can be described as coalgebras, and these specific sections are the coalgebras of the w-comonad K induced by χ. This simple observation suggests that the classical correspondence between categories with families and comprehension categories could be phrased within the framework of the correspondence between adjunctions and comonads. The structure-semantics adjunction [Dub70, Str72] can be used to show that comonads are 2-reflective in a suitable 2-category of adjunctions, where the 1-cells are pairs of functors commuting with the left adjoints. Of course, this reflection is in general far from being an equivalence. Nevertheless, we show that it lifts to a 2-reflection between generalised categories with families and w-comonads which becomes a biequivalence if one takes as morphisms of generalised categories with families functors that commute with left adjoints up to a natural vertical isomorphism. We call these *loose* morphisms. In type theoretic terms, this means preserving typing only up to (vertical) isomorphism. The equivalence in the discrete case is recovered thanks to the fact that vertical isomorphisms in discrete fibrations are identities.

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Barr exactness in classes of locally finite, transitive and reflexive Kripke frames

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Kripke frames (sets equipped with a binary relation) are one of the most popular semantics of modal logics (see [4] for a complete overview). They form the category $\mathbf{KFr}$, where the arrows are the so called *p-morphisms*. Images via p-morphisms are called *p-morphic images* and such images are *generated subframes* of their codomains. A Kripke frame $\mathcal{F}$ is called *locally finite* if, for each $p \in \mathcal{F}$, the smallest generated subframe containing p is finite (in literature, *image finite* Kripke frames are better known; locally finite Kripke frames are those Kripke frames whose transitive closure is image finite). We are interested in $\mathbf{KFr}_{lf}$, the full subcategory of *locally finite* Kripke frames: this subcategory is closed under coproducts (disjoint unions), generated subframes and p-morphic images. More generally, we are interested in any full subcategory $\mathcal{C} \subseteq \mathbf{KFr}_{lf}$ closed under the same operations (all colimits in $\mathcal{C}$ can be built from such operations). In [2], it has been shown that $\mathcal{C}$ is always *comonadic* over $\mathbf{Set}$.

The algebraic semantics of modal logic is given by *modal algebras*. In the so called Thomason duality [3], $\mathbf{KFr}_{lf}$ corresponds to $\mathrm{Pro}\mathbf{MA}_f$, the category of profinite modal algebras, with suitable morphisms, which is monadic over $\mathbf{Set}$ [2] (while image finite Kripke frames are dual to the topological modal algebras whose underlying topology is a Stone topology). Topological algebras and profiniteness are strictly related to classical problems such as canonical extensions of lattice-based algebras (among them are modal algebras). More generally, for any variety $\mathbf{V}$ of modal algebras generated by its finite members $\mathbf{V}_f$, the pro-completion [6] $\mathrm{Pro}\mathbf{V}_f$ is monadic over $\mathbf{Set}$. In the above duality, $\mathrm{Pro}\mathbf{V}_f$ corresponds to the class of locally finite Kripke frames validating the equations defining $\mathbf{V}$; the latter class has the aforementioned closure properties.

Our aim is to study categorical properties of classes of locally finite Kripke frames dual to $\mathrm{Pro}\mathbf{V}_f$, for some $\mathbf{V}$. In particular, we want to characterize regularity and Barr exactness, at least under the assumption that the Kripke frames are transitive. Indeed, it is possible to prove that: (i) such classes have all limits (being the ind-completion of the class of finite Kripke frames belonging to it [2]) and (ii) under the assumption of transitivity, the usual image factorization gives an (extremal epi, mono)-factorization. Therefore, to establish regularity, it only remains to check that extremal epimorphisms are stable under pullbacks. We present a partial solution for the reflexive and transitive case.

From now on, we fix a full subcategory $\mathcal{C}$ of *reflexive and transitive* locally finite Kripke frames closed under disjoint unions, generated subframes and p-morphic images. In this case, the stability of extremal epimorphisms under pullbacks can be rephrased in terms of the dual of the *amalgamation property*. A *co-amalgamation* for a finite family $f_1, \dots, f_n$ of epimorphisms with common codomain is a family $g_1, \dots, g_n$ of epimorphisms with common domain, such that all the compositions $f_i g_i$ exist and coincide. The category $\mathcal{C}$ is said to satisfy the *co-amalgamation property* if each finite family of epimorphisms with common codomain has a co-amalgamation.

Co-amalgamation can be used to find out necessary conditions for regularity (following the classification in [5, Section 6.3], see also [8, 7]): if $\mathcal{C}$ is regular, then it is forced to contain Kripke frames that can be built using co-amalgamation and p-morphic images.

The construction of a binary product in $\mathcal{C}$ can be performed by induction following the universal model construction, well known in the modal logic literature — see [1]. This implies that the product of a pair of objects in $\mathcal{C}'$ is a generated subframe of the product computed in any $\mathcal{C}$ containing $\mathcal{C}'$. The two products might coincide, for example, when $\mathcal{C}' = \mathcal{C} \cap \mathbf{Pos}_{lf}$, where $\mathbf{Pos}_{lf}$ is the class of locally finite posets. If this is the case, $\mathcal{C}'$ is closed under pullbacks in $\mathcal{C}$, being always closed under equalizers. This observation allows us to conclude that, if $\mathcal{C}$ is regular, then all its subclasses closed under finite products in $\mathcal{C}$ must be regular; in particular, $\mathcal{C} \cap \mathbf{Pos}_{lf}$ has to be regular, too. A case analysis, based on the co-amalgamation property, shows that exctly 8 subclasses of $\mathbf{Pos}_{lf}$ are regular. Therefore, the regular $\mathcal{C}$ must intersect $\mathbf{Pos}_{lf}$ in one of the 8 classes above; applying again the co-amalgamation property, we obtain 49 possible cases.

Barr exactness can also be studied. Similarly to what happens for regularity, given two regular $\mathcal{C}' \subseteq \mathcal{C}$, with $\mathcal{C}'$ closed under finite products in $\mathcal{C}$, if $\mathcal{C}$ is exact then $\mathcal{C}'$ is exact, too. In particular, $\mathcal{C} \cap \mathbf{Pos}_{lf}$ is exact if $\mathcal{C}$ is so. After having excluded a certain number of cases, we show that $\mathcal{C}$ is exact if it only contains the empty frame, or it is one of the following:

1. $\{\mathcal{F} \mid \mathrm{ht}(\mathcal{F}) \leq 1 \;\&\; \delta^e(\mathcal{F}) \leq 1\} \cong \mathbf{Set}$;
2. $\{\mathcal{F} \mid \mathrm{ht}(\mathcal{F}) \leq 1 \;\&\; \delta^e(\mathcal{F}) \leq 2\} \cong \mathbb{Z}_2^{+}$-$\mathbf{Set}$;
3. $\{\mathcal{F} \mid \mathrm{ht}(\mathcal{F}) \leq 2 \;\&\; \mathrm{wt}(\mathcal{F}) \leq 1 \;\&\; \delta^i(\mathcal{F}) \leq 1 \;\&\; \delta^e(\mathcal{F}) \leq 1\} \cong \mathbb{Z}_2^{\times}$-$\mathbf{Set}$;

Where ht and wt give bound for cardinality of chains, resp. antichains, and δ^e and δ^i give bound for cardinality of external, resp. internal clusters.

We are currently working on a full characterization of exactness in the reflexive and transitive case and on a generalization of this characterization without the reflexivity condition. In the latter context, exactness could be encountered in some non trivial cases. An example is given by the class $\mathbf{GL}$-Lin_{lf} of locally finite, transitive and irreflexive Kripke frames for which the restriction of the binary relation to each rooted generated subframe is a (irreflexive) linear order: $\mathbf{GL}$-Lin_{lf} is indeed equivalent to the category of presheaves $\mathbf{Set}^{(\mathbb{N},\leq)^{\mathrm{op}}}$.

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Inception Display Calculi

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In previous years, a formal connection between correspondence phenomena [7] and the theory of display calculi [1] was established, applying results and insights from unified correspondence theory [3]. One of the consequences of the aforementioned connection was the development of proper display calculi for the class of LE-logics [4], together with a method to convert a broad class of axioms (the class of all the analytic inductive inequalities) into rules that can be modularly added to the base calculus without disrupting the admissibility of the cut rule [5].

In this work we extend the framework of proper display calculi for LE-logics to include axiomatic extensions with axioms that are inductive [4] but not necessarily analytic inductive, greatly extending the class of axioms that can be converted into analytic rules. This class covers and properly extends all Sahlqvist axioms. A semantical analysis of the first-order correspondent of inductive inequalities suggests an approach that is similar in nature to that of Schroeder-Heister's Calculus of Higher-level Rules [8], and captures the whole acyclic portion of the substructural hierarchy [2], meaning that we can cope with arbitrary alternations of box-like and diamond-like connectives, as long as certain acyclicity conditions are satisfied.

Our approach is somewhat reminiscent of Negri's systems of rules [6], with the difference that no labelled G3c-like calculus is available for LE-logics, with the consequence that previously existing results cannot be applied to the case at hand. We make use of unified correspondence theory and the algorithm ALBA [4] to uniformly generate analytic rules for the previously mentioned inductive axiomatic extensions, and we call our new framework *Inception Display Calculus*.

Definition of the Inception Display Calculus framework

Inception Display calculi introduce special *side conditions* to the rules of proper display calculi.

Definition 0.1. Let $\mathcal{R}$ and $\mathcal{X}$ be a set of analytic structural rules (see [5]) and a set of structure variables, respectively. If $\Pi \vdash \Sigma$ is derivable using the rules of the base calculus together with $\mathcal{R}$, where $\Pi, \Sigma, \mathcal{R}$ may contain structure variables from $\mathcal{X}$, we write $[\Pi \vdash \Sigma]^{\mathcal{R}}_{\mathcal{X}}$ and we call it a *shallow contract*. A *shallow inception rule* is an analytic structural rule augmented with one or more shallow contracts as side conditions, namely a rule of the following form:

$$\frac{X_1 \vdash Y_1 \quad \cdots \quad X_n \vdash Y_n \qquad [\Pi_1 \vdash \Sigma_1]^{\mathcal{R}_1}_{\mathcal{X}_1} \quad \cdots \quad [\Pi_m \vdash \Sigma_m]^{\mathcal{R}_m}_{\mathcal{X}_m}}{X \vdash Y}$$

Sometimes we write $[\pi]^{\mathcal{R}}_{\mathcal{X}}$ in place of $[\Pi \vdash \Sigma]^{\mathcal{R}}_{\mathcal{X}}$, where π is a derivation of $\Pi \vdash \Sigma$, omitting subscripts and superscripts when they are clear from the context.

Definition 0.2. Let us define inductively *depth-n inception rules* ($n \geq 0$) and *depth-n contracts* ($n \geq 1$).

- Depth-0 inception rules are the analytic structural rules; depth-1 contracts are the shallow contracts and depth-1 inception rules are the shallow inception rules.
- Suppose we defined depth-k inception rules and contracts for every $k < n$, for a certain $n > 1$. A depth-n contract is a side condition of the form $[\Pi \vdash \Sigma]^{\mathcal{R}}_{\mathcal{X}}$, where $\mathcal{R}$ is a set of inception rules of depth smaller than n. A depth-n inception rule is an analytic structural rule augmented with contracts of depth not greater than n.

An *inception rule* (resp. a *contract*) is a depth-n inception rule (resp. depth-n contract) for some $n \geq 1$. We also say that its depth is n. A derivation has *finite dreams* if it contains a finite number of instances of contracts.

In this work, we consider only derivations with finite dreams. As an example, consider the inductive but not analytic inductive axiom $\Box(\Diamond p \circ p) \circ \Diamond p \vdash p$.

$$
\begin{array}{ll}
 & \text{ALBA run computing the inception rule for } \Box(\Diamond p \circ p) \circ \Diamond p \vdash p\text{:} \\
\hline
 & \Box(\Diamond p \circ p) \circ \Diamond p \leq p \\
\text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} [\mathbf{i} \leq \Box(\Diamond p \circ p) \ \&\ \mathbf{j} \leq p \ \&\ p \leq \mathbf{m} \Rightarrow \mathbf{i} \circ \Diamond \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} [\mathbf{i} \leq \Box(\Diamond \mathbf{m} \circ \mathbf{m}) \ \&\ \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \circ \Diamond \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} [\forall \mathbf{n} (\Diamond \mathbf{m} \circ \mathbf{m} \leq \mathbf{n} \Rightarrow \mathbf{i} \leq \Box \mathbf{n}) \ \&\ \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \circ \Diamond \mathbf{j} \leq \mathbf{m}] \\
\text{iff} & \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} [\forall \mathbf{n} (\forall \mathbf{k} \forall \mathbf{h} (\mathbf{k} \leq \mathbf{m} \ \&\ \mathbf{h} \leq \mathbf{m} \Rightarrow \Diamond \mathbf{k} \circ \mathbf{h} \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \Box \mathbf{n}) \ \&\ \mathbf{j} \leq \mathbf{m} \Rightarrow \mathbf{i} \circ \Diamond \mathbf{j} \leq \mathbf{m}]
\end{array}
$$

The last line of the derivation above gives us the first-order correspondent of $\Box(\Diamond p \circ p) \circ \Diamond p \vdash p$, from which we can obtain the depth-1 inception rule

$$R_0 \, \frac{Y \vdash Z \qquad [X \vdash \check{\Box} N]^{\mathcal{R}}_{\{N\}}}{X \,\hat{\circ}\, \hat{\Diamond} Y \vdash Z}$$

where $\mathcal{R}$ is the singleton containing

$$R_1 \, \frac{K \vdash Z \qquad H \vdash Z}{\hat{\Diamond} K \,\hat{\circ}\, H \vdash N} \; .$$

We show how to derive the axiom $\Box(\Diamond p \circ p) \circ \Diamond p \vdash p$ from the rule just obtained, where adjunction rules are omitted for brevity:

$$
R_0 \, \cfrac{\cfrac{\cfrac{p \vdash p \qquad [\pi]^{\mathcal{R}}_{\{N\}}}{\Box(\Diamond p \circ p) \,\hat{\circ}\, \hat{\Diamond} p \vdash p}}{\Box(\Diamond p \circ p) \,\hat{\circ}\, \Diamond p \vdash p}}{\Box(\Diamond p \circ p) \circ \Diamond p \vdash p} \text{, where } \pi \text{ is: } \quad
R_1 \, \cfrac{\cfrac{\cfrac{\cfrac{p \vdash p \qquad p \vdash p}{\hat{\Diamond} p \,\hat{\circ}\, p \vdash W}}{\Diamond p \,\hat{\circ}\, p \vdash W}}{\Diamond p \circ p \vdash W}}{\Box(\Diamond p \circ p) \vdash \check{\Box} W}
$$

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Topoi with enough points

Ivan Di Liberti and Morgan Rogers

General landscape

This paper is concerned with the general problem of assessing whether a topos has enough points, with motivations coming both from geometry and logic. This problem has been influential in both fields up to the present day (see for example the work of Lurie [Lur04, VII, 4.1] and Barwick et al. [BGH18, II.3], for the case of geometry, and Espíndola and Kanalas [EK23] for the case of logic). We begin with an account of the development of this problem to properly frame our contribution and the significance of our work.

In 1964 Pierre Deligne proved a very celebrated theorem in topos theory.

Theorem (Deligne, [BDSD06, Exposé VI, 9.0])**.** Every locally coherent topos has enough points.

The theorem's original motivation came from algebraic geometry, but after Joyal and Reyes developed the theory of classifying topoi, it was observed by Lawvere that Deligne's theorem was essentially the statement of Gödel's completeness theorem for first order logic in disguise. This realisation crowned Deligne's theorem as a major result in categorical logic, and a source of inspiration for finding other completeness-like results using techniques from topos theory. To the present day Deligne's theorem remains the main argument to show that a wide class of topoi have enough points, and to some extent this paper investigates the limits (and the possibly unexploited potential) of this result.

Following Deligne's theorem, new results eventually emerged showing that further classes of topoi have enough points. Makkai and Reyes proved that *separable* topoi have enough points.

Theorem (Makkai and Reyes, [MR06, Theorem 6.2.4, page 180])**.** Let $\mathcal{C}$ be a countable category with pullbacks and J a Grothendieck topology generated by a countable family of sieves. Then $\mathrm{Sh}(\mathcal{C}, J)$ has enough points.

This result was inspired by the Fourman-Grayson completeness theorem for the logic L_{ω_1,ω_0} (see [FG82]), and indeed it is almost the translation of it into topos-theoretic language through the bridge of classifying topoi. The proof in [MR06] is a bit sketchy, and of model theoretic inspiration, thus it is hard to compare this result to Deligne's.

Recent developments

Lurie has imported Deligne's original argument to the world of ∞-topoi [Lur04, VII, 4.1]. The proof carries with minor adjustments, and under the mild additional assumption that the ∞-topos is hypercomplete. On the logical side, the main advances are due to Espíndola [Esp19, Esp20, EK23]; the most recent categorical analysis with Kanalas delivers a vast generalization of the original results achieved in Espíndola's PhD thesis.

Simultaneously to these developments, the topos theory community has been trying to understand the limits of Deligne's original argument and its possible generalization. Quite independently the authors of this paper and Tim Campion conjectured that every *locally finitely presentable topos could have enough points.* This conjecture finds its motivations in a number of examples, including the fact that coherent topoi are locally finitely presentable ([Joh02, D3.3.12]); presheaf topoi are often not coherent and yet they are always locally finitely presentable, and they are the easiest example of topos with enough points, and finally Hoffmann-Lawson duality

[HL78] from the theory of locales seems to support the conjecture that exponentiable topoi (which include locally finitely presentable topoi) have enough points. Before discussing our contribution, we should say that we did not manage to prove (or disprove) the conjecture. Yet our analysis seems to suggest that these methods cannot be sufficient to deliver a proof of the conjecture, if any such exists.

Our contribution

We employ the notion of collage in our presentation and generalization of Deligne's proof. Diagrams in the collage offer a convenient framework where points (on the left) can interact with objects (on the right) of a topos as if they were in the same category.

$$\begin{tikzcd} p \arrow[r, "z"] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \end{tikzcd}$$

After introducing the notion of *improvement*, designed to isolate the central idea of Deligne's proof, we prove our main theorem (which we express in a simplified, easy-to-read form).

Theorem. Let $j : \mathcal{F} \rightarrowtail \mathcal{E}$ be an inclusion of toposes. Suppose that for every point p admits an improvement. If $\mathcal{E}$ has enough points, then $\mathcal{F}$ has enough points.

We show that our refinement of Deligne's argument can be used to recover every existing result of this kind (for 1-topoi), including the most recent ones about κ-coherent κ-topos. Our strategy allows us to relax the assumptions on the site so that one is no longer required to control the cardinality of the set of morphisms.

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Multi-type universal algebra: categorical equivalence *

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Multi-type algebras are special kinds of heterogeneous algebras [1] which can be represented as the corresponding single-type algebras and vice versa. The motivation for studying this kind of mathematical structure comes from algebraic proof theory. In [4], it is showed that the algorithm ALBA can be used to transform analytic inductive formulas into rules of display calculus which satisfy the premises of Belnap's cut-elimination theorem. However, there are many important logics whose axioms are not analytic inductive, and cut-free display calculus can not be provided for them by using ALBA directly. In order to settle this problem, the multi-type methodology is introduced which represents single-type axiomatizations as multi-type axiomatizations where axioms in the multi-type languages are all analytic inductive formulas. This methodology has been applied successfully to semi De Morgan logic [2], linear logic [5], logic of bilattices [3] and so on, whose axioms are not analytic inductive. In this paper, we introduce general definitions for both single-type and multi-type algebras and homomorphisms on them. We define a functor from the category of multi-type algebras to the category of single-type algebras and vice versa and show categorical equivalence of those two categories.

We first introduce the general definition of multi-type algebras and homomorphisms on them.

A *multi-type algebraic language* is a tuple $\mathfrak{L} = (\mathcal{L}_1, \mathcal{L}_2, h, e)$ where $\mathcal{L}_1$ and $\mathcal{L}_2$ are sets of function symbols and h and e are two unary function symbols. A *multi-type* $\mathfrak{L}$-algebra is a tuple $\mathfrak{A} = (\mathbf{A}, \mathbf{B}, h, e)$ such that: (1) $\mathbf{A}$ is an $\mathcal{L}_1$-algebra and $\mathbf{B}$ is an $\mathcal{L}_2$-algebra. (2) $h : \mathbf{A} \to \mathbf{B}$ is a unary surjective map and $e : \mathbf{B} \to \mathbf{A}$ is a unary injective map. (3) For any $\alpha_1, \cdots, \alpha_n \in \mathbf{B}$ and n-ary $f \in \mathcal{L}_1 \cap \mathcal{L}_2$, $f^{\mathbf{B}}(\alpha_1, \cdots, \alpha_n) = hf^{\mathbf{A}}(e(\alpha_1), \cdots, e(\alpha_n))$. Let $\mathfrak{A}_1 = (\mathbf{A}_1, \mathbf{B}_1, h_1, e_1)$ and $\mathfrak{A}_2 = (\mathbf{A}_2, \mathbf{B}_2, h_2, e_2)$ be any multi-type $\mathfrak{L}$-algebras. A *multi-type homomorphism* between $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is a pair $F = (F^{\mathbf{A}}, F^{\mathbf{B}})$ of maps such that: (1) $F^{\mathbf{A}} : \mathbf{A}_1 \to \mathbf{A}_2$ and $F^{\mathbf{B}} : \mathbf{B}_1 \to \mathbf{B}_2$ are homomorphisms. (2) $F^{\mathbf{B}} \circ h_1 = h_2 \circ F^{\mathbf{A}}$ and $F^{\mathbf{A}} \circ e_1 = e_2 \circ F^{\mathbf{B}}$.

Now we are ready to define the single-type representation of multi-type algebras and the multi-type representation of single-type algebras.

Definition 1. *Let $\mathfrak{L} = (\mathcal{L}_1, \mathcal{L}_2, h, e)$ be a multi-type algebraic language and $\mathcal{L}_3 = (\mathcal{L}_2 - \mathcal{L}_1)$, the* single-type representation *of $\mathfrak{L}$ is $\mathfrak{L}_+ = \mathcal{L}_1 \cup \mathcal{L}_3 \cup \{e \circ h\}$. Given a multi-type $\mathfrak{L}$-algebra $\mathfrak{A} = (\mathbf{A}, \mathbf{B}, h, e)$, the* single-type representation *of $\mathfrak{A}$ is $\mathfrak{A}_+ = (\mathbf{A}, \{f^{\mathfrak{A}_+} \mid f \in \mathcal{L}_3\}, e\circ h)$ such that for any n-ary $f_i \in \mathcal{L}_3$ and $a, a_1, \cdots, a_n \in A$, $f_i^{\mathfrak{A}_+}(a_1, \cdots, a_n) := ef_i^{\mathbf{B}}(h(a_1), \cdots, h(a_1))$. Let $\mathfrak{A}_1 = (\mathbf{A}_1, \mathbf{B}_1, h, e)$ and $\mathfrak{A}_2 = (\mathbf{A}_2, \mathbf{B}_2, h, e)$ be any multi-type $\mathfrak{L}$-algebras and $F = (F^{\mathbf{A}}, F^{\mathbf{B}}) : \mathfrak{A}_1 \to \mathfrak{A}_2$ be any multi-type homomorphism, the* single-type representation *of F is $F_+ : \mathfrak{A}_{1+} \to \mathfrak{A}_{2+}$ defined by $F_+(a) := F^{\mathbf{A}}(a)$ for any $a \in A_1$.*

Given a single-type algebra, there are different ways of representing it as a multi-type algebra which depends on which operators we want to keep, to destroy or to rebuild on kernels. In order to divide our algebraic language properly, we introduce the notion of parameter.

Definition 2. *Let $\mathcal{L}$ be an algebraic language. A* parameter *on $\mathcal{L}$ is a tuple $\mathsf{P} = (\mathcal{L}_1, \mathcal{L}_2, \sigma)$ such that $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{\sigma\}$ and $\mathcal{L}_1$, $\mathcal{L}_2$ and $\{\sigma\}$ are pairwise disjoint. An $\mathcal{L}$-algebra $\mathbf{C}$ is called*

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P-algebra *if for any* $a, a_1, \cdots, a_n \in \mathbf{C}$ *and n-ary* $f \in \mathcal{L}$, $\sigma\sigma(a) = a$ *and* $\sigma f(\sigma(a_1), \cdots, \sigma(a_n)) = f(a_1, \cdots, a_n)$.

Now we are ready to give the multi-type representation of a single-type algebra relative to a parameter.

Definition 3. *Let* $\mathcal{L}$ *be an algebraic language,* $\mathsf{P} = (\mathcal{L}_1, \mathcal{L}_2, \sigma)$ *be a parameter on* $\mathcal{L}$ *and* $\mathbf{C}$ *be any* P*-algebra. The* multi-type representation *of* $\mathcal{L}$ *relative to* P *is* $\mathcal{L}^+ = (\mathcal{L}_1, \mathcal{L}_2, h, e)$. *The* multi-type representation *of* $\mathbf{C}$ *is* $\mathbf{C}^+ = (\mathbf{A}, \mathbf{B}, h, e)$ *such that: (1)* $\mathbf{A}$ *is the* $\mathcal{L}_1$*-reduct of* $\mathbf{C}$. *(2)* $h : \mathbf{A} \to \mathbf{B}$ *and* $e : \mathbf{B} \to \mathbf{A}$ *such that for any* $a \in A$ *and* $\alpha \in \sigma(A)$, $h(a) := \sigma(a)$ *and* $e(\alpha) := \alpha$. *(3)* $\mathbf{B} = (\sigma[A], \{f^{\mathbf{B}} \mid f \in \mathcal{L}\})$ *such that for any n-ary* $f \in \mathcal{L}$ *and* $\alpha_1, ...\alpha_n \in \sigma[A]$, $f^{\mathbf{B}}(\alpha_1, ..., \alpha_n) := hf^{\mathbf{C}}(e(\alpha_1), ..., e(\alpha_n))$. *Let* $\mathbf{C}_1, \mathbf{C}_2$ *be any* $\mathcal{L}$*-algberas and* $F : \mathbf{C}_1 \to \mathbf{C}_2$ *be any homomorphism. Let* ${\mathbf{C}_i}^+ = (\mathbf{A}_i, \mathbf{B}_i, h_i, e_i)$ *for* $i \in \{1, 2\}$. The multi-type representation *of* F *is* $F^+ = (F^{\mathbf{A}}, F^{\mathbf{B}})$ *from* ${\mathbf{C}_1}^+$ *to* ${\mathbf{C}_2}^+$, *where* $F^{\mathbf{A}} : \mathbf{A}_1 \to \mathbf{A}_2$ *and* $F^{\mathbf{B}} : \mathbf{B}_1 \to \mathbf{B}_2$ *such that for any* $a \in \mathbf{A}_1$ *and* $\alpha \in \mathbf{B}_1$, $F^{\mathbf{A}}(a) := F(a)$ *and* $F^{\mathbf{B}}(\alpha) := h_2 \circ F \circ e_1(\alpha)$.

Now we are ready to establish the categorical equivalence between single-type and multi-type algebras.

Given a multi-type algebraic language $\mathfrak{L} = (\mathcal{L}_1, \mathcal{L}_2, h, e)$, $\mathsf{Cat}_{\mathfrak{L}}$ denotes the category of multi-type $\mathfrak{L}$-algebras and multi-type homomorphisms between them, and $\mathsf{Cat}_{\mathfrak{L}_+}$ denotes the category of P-algebras and homomorphisms between them, where $\mathsf{P} = (\mathcal{L}_1, \mathcal{L}_2, e \circ h)$ is the parameter on algebraic language $\mathfrak{L}_+$. $\mathsf{S}_{\mathfrak{L}} : \mathsf{Cat}_{\mathfrak{L}} \to \mathsf{Cat}_{\mathfrak{L}_+}$ is a functor such that for any $\mathfrak{A}, F \in \mathsf{Cat}_{\mathfrak{L}}$, $\mathsf{S}_{\mathfrak{L}}(\mathfrak{A}) := \mathfrak{A}_+$ and $\mathsf{S}_{\mathfrak{L}}(F) := F_+$. $\mathsf{M}_{\mathfrak{L}} : \mathsf{Cat}_{\mathfrak{L}_+} \to \mathsf{Cat}_{\mathfrak{L}}$ is a functor such that for any $\mathbf{C}, F \in \mathsf{Cat}_{\mathfrak{L}_+}$, $\mathsf{M}_{\mathfrak{L}}(\mathbf{C}) := \mathbf{C}^+$ and $\mathsf{M}_{\mathfrak{L}}(F) := F^+$.

Given an algebraic language $\mathcal{L}$ and a parameter $\mathsf{P} = (\mathcal{L}_1, \mathcal{L}_2, \sigma)$ on $\mathcal{L}$. $\mathsf{Cat}_{\mathcal{L}}$ denotes the category of P-algebras and homomorphisms between them, and $\mathsf{Cat}_{\mathcal{L}^+}$ denotes the category of multi-type $\mathcal{L}^+$-algebras and multi-type homomorphisms between them. $\mathsf{M}_{\mathcal{L}} : \mathsf{Cat}_{\mathcal{L}} \to \mathsf{Cat}_{\mathcal{L}^+}$ is a functor such that for any $\mathbf{C}, F \in \mathsf{Cat}_{\mathcal{L}}$, $\mathsf{M}_{\mathcal{L}}(\mathbf{C}) := \mathbf{C}^+$ and $\mathsf{M}_{\mathcal{L}}(F) := F^+$. $\mathsf{S}_{\mathcal{L}} : \mathsf{Cat}_{\mathcal{L}^+} \to \mathsf{Cat}_{\mathcal{L}}$ is a functor such that for any $\mathfrak{A}, F \in \mathsf{Cat}_{\mathcal{L}^+}$, $\mathsf{S}_{\mathcal{L}}(\mathfrak{A}) := \mathfrak{A}_+$ and $\mathsf{S}_{\mathcal{L}}(F) := F_+$.

According to definitions above, we can prove categorical equivalence between category of single-type algebras and category of multi-type algebras as stated the following theorem.

Theorem 1. *Let* $\mathfrak{L}$ *be a multi-type algebraic language, then* $\mathsf{S}_{\mathfrak{L}} : \mathsf{Cat}_{\mathfrak{L}} \to \mathsf{Cat}_{\mathfrak{L}_+}$ *and* $\mathsf{M}_{\mathfrak{L}} : \mathsf{Cat}_{\mathfrak{L}_+} \to \mathsf{Cat}_{\mathfrak{L}}$ *forms a categorical equivalence between them. Let* $\mathcal{L}$ *be an algebraic language and* P *be a parameter on* $\mathcal{L}$, *then* $\mathsf{M}_{\mathcal{L}} : \mathsf{Cat}_{\mathcal{L}} \to \mathsf{Cat}_{\mathcal{L}^+}$ *and* $\mathsf{S}_{\mathcal{L}} : \mathsf{Cat}_{\mathcal{L}^+} \to \mathsf{Cat}_{\mathcal{L}}$ *forms a categorical equivalence between them.*

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Multi-type universal algebra: Transfer of properties *

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Multi-type algebras are special type of *heterogeneous algebras* used in previous research to provide multi-type algebraic semantics for logics such as Semi De Morgan logic [2], and linear logic [3]. This class of algebras dually corresponds to the multi-type logic equivalent to the single-type logic under consideration. These studies were conducted to overcome some problematic issues related to the proof-theoretic perspective on those given logic systems which are not properly displayable in the sense of Wansing [5]. The present paper aims to capture the general construction of multi-type algebras and to study their universal algebraic properties.

Definition 1. *A multi-type algebra $\mathbb{H}$ is a tuple $(\mathbb{A}, \mathbb{K}, h, e)$ such that*
1. $\mathbb{A}$ and $\mathbb{K}$ are some (single-type) algebras.
2. $h : \mathbb{A} \to \mathbb{K}$ is a surjective map and $e : \mathbb{K} \to \mathbb{A}$ is an injective map.
3. The set of operations $\{f^{\mathbb{K}} \mid f \in \mathcal{O}_{\mathbb{A}}\}$ is a subset of the basic operations $\mathcal{O}_{\mathbb{K}}$ on the kernel $\mathbb{K}$, where for any n-ary operation $f \in \mathcal{O}_{\mathbb{A}}$, and $\alpha_1, \alpha_2, \cdots, \alpha_n \in K$,

$$f^{\mathbb{K}}(\alpha_1, \alpha_2, \cdots, \alpha_n) = hf(e(\alpha_1), e(\alpha_2), \cdots, e(\alpha_n)).$$

Given any multi-type algebras $\mathbb{H}_1 = (\mathbb{A}_1, \mathbb{K}_1, h_1, e_1)$ and $\mathbb{H}_2 = (\mathbb{A}_2, \mathbb{K}_2, h_2, e_2)$, a multi-type homomorphism between $\mathbb{H}_1$ and $\mathbb{H}_2$ is a pair $F = (F^{\mathbb{A}}, F^{\mathbb{K}})$, such that $F^{\mathbb{A}} : \mathbb{A}_1 \to \mathbb{A}_2$ and $F^{\mathbb{K}} : \mathbb{K}_1 \to \mathbb{K}_2$ are homomorphisms of these algebras as single-type algebras, and $F^{\mathbb{K}} \circ h_1 = h_2 \circ F^{\mathbb{A}}$ and $F^{\mathbb{A}} \circ e_1 = e_2 \circ F^{\mathbb{K}}$. We define multi-type representation of a single-type algebra as follows.

Definition 2. *Let $\mathcal{K}$ be a class of algebra with set of the basic operations $\mathcal{O}$ on the algebras in $\mathcal{K}$. Let $\sigma \in \mathcal{O}$ be a unary operation on the algebras in $\mathcal{K}$. Suppose there exist disjoint set of the basic operations $\mathcal{O}_1$, $\mathcal{O}_2$, and $\{\sigma\}$ such that $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \{\sigma\} = \mathcal{O}$. Then the multi-type representation of any $\mathbb{A} \in \mathcal{K}$, $Mult_{(\sigma, \mathcal{O}_1, \mathcal{O}_2)}(\mathbb{A}) = (\mathbb{A}', \mathbb{K}, h, e)$, where*
1. $\mathbb{A}'$ is an $\mathcal{O}_1$-reduct of $\mathbb{A}$,
2. $\mathbb{K} = (\sigma(A), \{f^{\mathbb{K}} \mid f \in \mathcal{O}\})$,
3. $h : \mathbb{A}' \to \mathbb{K}$ and $e : \mathbb{K} \to \mathbb{A}'$ are surjective and injective maps respectively, such that for any $\alpha \in \sigma(A)$, $e(\alpha) = \alpha$ and $e \circ h = \sigma$.

In Definition 2, the algebra $\mathbb{K}$ is called the kernel of the multi-type representation of $\mathbb{A}$. We also define the multi-type representation of a homomorphism between two single-type algebras $F : \mathbb{A}_1 \to \mathbb{A}_2$ as $Mult_{(\sigma, \mathcal{O}_1, \mathcal{O}_2)}(F) = (F^{\mathbb{A}}, F^{\mathbb{K}})$ where $F^{\mathbb{K}} : \mathbb{K}_1 \to \mathbb{K}_2$ and $F^{\mathbb{A}} : \mathbb{A}_1 \to \mathbb{A}_2$ such that for any $\alpha \in \mathbb{K}_1$, and $a \in \mathbb{A}_1$, $F^{\mathbb{K}}(\alpha) = h_2 F e_1(\alpha)$ and $F^{\mathbb{A}}(a) = F(a)$. For a given class of single-type algebras acting as algebraic semantics, this construction allows us to define a class of multi-type algebras that serve as multi-type algebraic semantics for the same logic. It can be shown that any category of single-type algebras is categorically equivalent to the category of multi-type algebras defined from it by the above construction.

The multi-type construction transfers many universal algebraic properties of the single-type algebra $\mathbb{A}$ to the kernel of its multi-type representation under certain conditions. In this work, we investigate this phenomenon and show several properties which are transferred in this manner. Table 1 depicts some properties of the kernels and corresponding conditions on the single-type

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	Property in $\mathbb{K}$	Conditions on $\mathbb{A}$	Side conditions
1	Distributivity	$\sigma(\sigma a \wedge \sigma(\sigma b \vee \sigma c)) = \sigma(\sigma(\sigma a \wedge \sigma b) \vee \sigma(\sigma a \wedge \sigma c))$	-
2	$f^{\mathbb{K}}(\alpha \sqcup^{\epsilon} \beta) = f^{\mathbb{K}}(\alpha) \sqcup f^{\mathbb{K}}(\beta)$	$f(\sigma a \vee^{\epsilon} \sigma(b)) = \sigma(f(a) \vee f(b))$	$\sigma f(\sigma a) = f(a)$
3	$f^{\mathbb{K}}(\alpha \sqcap^{\epsilon} \beta) = f^{\mathbb{K}}(\alpha) \sqcap f^{\mathbb{K}}(\beta)$	$f(\sigma a \wedge^{\epsilon} \sigma(b)) = \sigma(f(a) \wedge f(b))$	$\sigma f(\sigma a) = f(a)$
4	Injective (surjective) homomorphism	Injective (surjective) homomorphism	-
5	Congruence permutable	congruence permutable	-
6	Congruence distributive	congruence distributive	-
7	Arithmetical	arithmetical	-
8	Amalgamation property	amalgamation property	$H(\mathcal{V}') \subseteq Ker(H(\mathcal{V}))$
9	Superamalgamation property	superamalgamation property	$H(\mathcal{V}') \subseteq Ker(H(\mathcal{V}))$

Table 1: Properties of kernels and corresponding condition on single type algebras. The $\epsilon \in \{\partial, 1\}$ where $\vee^{\epsilon} = \vee$ and $\sqcup^{\epsilon} = \sqcup$ when $\epsilon = 1$, and $\vee^{\epsilon} = \wedge$ and $\sqcup^{\epsilon} = \sqcap$ when $\epsilon = \partial$. Similarly, $\wedge^{\epsilon} = \wedge$ and $\sqcap^{\epsilon} = \sqcap$ when $\epsilon = 1$, and $\wedge^{\epsilon} = \vee$ and $\sqcap^{\epsilon} = \sqcup$ when $\epsilon = \partial$.

algebras along with some side conditions on operator σ which imply the given property for the kernel. We have a particular interest in algebras with (semi) lattice structure with monotone and idempotent unary operator σ, as these algebras often provide algebraic semantics for commonly studied logics. Under this assumption, we define an order $\leq_{\mathbb{K}}$ on the kernel as follows: for any $\alpha, \beta \in \mathbb{K}$, $\alpha \leq_{\mathbb{K}} \beta$ iff $e(\alpha) \leq e(\beta)$ where $\leq$ is the standard order on $\mathbb{A}$. Under the order $\leq_{\mathbb{K}}$, the algebra $\mathbb{K}$ forms a lattice with join and meet defined as $\alpha \sqcup \beta := h(e(\alpha) \vee e(\beta))$ and $\alpha \sqcap \beta := h(e(\alpha) \wedge e(\beta))$, respectively. The first three items in Table 1 relate order theoretic-properties of single-type algebras with lattice structure to those of their kernels. The fourth item relates properties of single-type and kernel homomorphisms. The remaining items pertain to the case when the class of single-type algebras $\mathcal{V}$ and the class of their kernels $\mathcal{V}'$ form varieties. For items 5 to 7 by Malcev Conditions [1], these properties are equivalent to the existence of corresponding Malcev terms. These items are proven by showing that the existence of Malcev terms on $\mathcal{V}$ implies the existence of those terms on $\mathcal{V}'$. For items 8 and 9, $H(\mathcal{V})$ and $H(\mathcal{V}')$ denote the sets of all homomorphisms on $\mathcal{V}$ and $\mathcal{V}'$, respectively, and the map Ker assigns a $\mathcal{V}$-homomorphism to the kernel component of its multi-type representation. Amalgamation and superamalgamation properties are important because they dually correspond to certain interpolation properties [4] of the logic of the given variety.

Moreover, any congruence relation θ on $\mathbb{A}$ defines a congruence relation $\theta_{\mathbb{K}}$ on $\mathbb{K}$, defined by $\alpha\theta_{\mathbb{K}}\beta$ iff $e(\alpha)\theta e(\beta)$. This defines a lattice homomorphism κ between $Con(\mathbb{A})$ and $Con(\mathbb{K})$ given by $\kappa : Con(\mathbb{A}) \to Con(\mathbb{K})$ which assigns any $\theta \in Con(\mathbb{A})$ to $\kappa(\theta) = \theta_{\mathbb{K}}$. This relationship between congruence lattices is important in studying conditions under which the class of kernels defined by a variety of single-type algebras, forms a variety of multi-type algebras.

For future direction, we would like to investigate whether the results mentioned above can be derived from the properties of functor $Ker_{(\sigma,\mathcal{O}_1,\mathcal{O}_2)}$, which assigns a single-type algebra to kernel of its multi-type representation and a single-type homomorphism to the kernel component of its multi-type representation.

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Conditional Esakia Duality

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Classical conditional logic strives to resolve issues that mathematicians and philosophers alike have had with traditional implication. We often have an intuition that $A \to B$ should speak to a connection between A and B — they should not be irrelevant or coincidentally connected. Weiss developed **ICK** from Chellas classical conditional in [1] by adding conditional implication, $\boxright$, to basic intuitionistic logic. The conditional intuitively represents a stricter interpretation of implication — $\varphi \boxright \psi$ is only the case if φ is relevant to ψ. This intuition is reflected in the frame semantics of **ICK**.

Definition 1. An *ICK-frame* is a tuple $(X, \leq, f)$ where $(X, \leq)$ is a preorder and

$$f : X \times Up(X, \leq) \to Up(X, \leq)$$

is a *selection function* such that $x \leq y$ implies $f(y, a) \subseteq f(x, a)$ for all $a \in Up(X, \leq)$. Proposition letters are interpreted via a valuation which assigns an upset of $(X, \leq)$ to each proposition letter, and $\wedge, \vee, \neg$ and $\to$ are interpreted as usual. For $x \in X$ we let

$$x \Vdash \varphi \boxright \psi \quad \text{iff} \quad f(w, [\![\varphi]\!]) \subseteq [\![\psi]\!],$$

where $[\![\varphi]\!] = \{w \in X \mid w \Vdash \varphi\}$.

We can think of f as picking out the worlds relevant to φ at w. While a selection function best approximates our conditional motivations, it is sometimes easier to view it as an upset-indexed a family of relations $\{R_\alpha\}$, where $(\leq \circ R_\alpha \circ \leq) \subseteq R_\alpha$ for each relation. This allows us to view each relation as a modal relation in the sense of intuitionistic normal modal logic [4].

The new connective $\boxright$ can be axiomatised by adding to intuitionistic logic the axioms

$$(\varphi \boxright (\psi \wedge \theta)) \leftrightarrow ((\varphi \boxright \psi) \wedge (\varphi \boxright \theta)) \qquad \text{and} \qquad (\varphi \boxright \top) \leftrightarrow \top$$

and congruence rules, resulting in the logic **ICK**. We find the following algebraic semantics:

Definition 2. A *conditional Heyting algebra* is a tuple $(\mathcal{A}, \boxright)$ consisting of a Heyting algebra $\mathcal{A}$ and a binary operator $\boxright$ satisfying $a \boxright (b \wedge c) = (a \boxright b) \wedge (a \boxright c)$ and $a \boxright 1 = 1$.

Inspired by the duality for intuitionistic normal modal logic [4, 2] and Weiss' work on **ICK** [3], we define topologised frame semantics as follows.

Definition 3. A *conditional Esakia space* is an Esakia space $\mathbb{X} = (X, \leq, \tau)$ equipped with a family of point-closed relations $\{R_A \mid A \in ClpUp(\mathbb{X})\}$ such that for each $A, B \in ClpUp(\mathbb{X})$:

$$\Box_{R_A}(B) := \{x \in X \mid R_A[x] \subseteq B\} \in ClpUp(\mathbb{X}) \qquad \text{and} \qquad (\leq \circ R_A \circ \leq) = R_A.$$

It is well known that collection of clopen upsets of an Esakia space $\mathbb{X}$ forms a Heyting algebra, denoted by $\mathbb{X}^+$. We can obtain a conditional Heyting algebra $(\mathbb{X}^+, \boxright)$ by defining

$$A \boxright B = \Box_{R_A}(B).$$

Conversely, every Heyting algebra $\mathcal{A}$ corresponds to an Esakia space $\mathcal{A}_+$ based on the prime filters of $\mathcal{A}$. In particular, we know that every clopen upset of $\mathcal{A}_+$ is of the form $\tilde{a} := \{x \in \mathcal{A}_+ \mid a \in x\}$ for some $a \in \mathcal{A}$. Ergo, the following definition

$$xR_{\tilde{a}}y \qquad \text{iff} \qquad \{b \in \mathcal{A} \mid a \mathrel{\Box\!\!\rightarrow} b \in x\} \subseteq y,$$

results in a conditional Esakia space $(\mathcal{A}_+, \{R_{\tilde{a}}\})$. In fact, the assignments above give rise to a dual equivalence between categories:

Theorem 4. *The category of conditional Esakia spaces (with suitable morphisms) is dually equivalent to the category of conditional Heyting algebras and homomorphisms.*

This duality allows us to prove several frame completeness results. Beginning with **ICK** we note that if **ICK** $\nvdash \varphi$ then there exists a conditional Heyting algebra such that $\mathcal{A} \not\models \varphi$. Hence there exists a conditional Esakia space $\mathbb{X} = (X, \leq, \tau, \{R_\alpha\})$ and a valuation V such that $(\mathbb{X}, V) \nVdash \varphi$. Forgetting the topology *almost* gives an ICK-frame, except it lacks relations R_α when α is a non-clopen upset. We can fill in these missing relations by setting:

$$S_\alpha = \begin{cases} R_\alpha & \text{if } \alpha \text{ is a clopen upset} \\ \emptyset & \text{otherwise} \end{cases}$$

Since the clopen-indexed relations are unchanged we find that $((X, \leq, \{S_\alpha\}), V) \nVdash \varphi$, so that:

Theorem 5. *The logic* **ICK** *is sound and complete with respect to ICK-frames.*

This example highlights the two prongs of a duality completeness proof. When we extend **ICK** with a collection of axioms we induce both a *frame correspondence condition* and a *counter-example space*. We then need to extend the space to a frame which satisfies this correspondence condition for all upsets while also leaving the clopen relations (which guarantee the counter-example) unchanged. We call these "nice extensions" *fill-ins*, since we are in a sense filling in the missing upset relations on the underlying frame of a conditional Esakia space. Below we list several extensions of **ICK** together with their frame correspondent and (one possible) fill-in that can be used to prove completeness.

Axiom	Frame Condition	Fill-in
$\varphi \mathrel{\Box\!\!\rightarrow} \varphi$	$R_A[x] \subseteq A$	$R_p[x] \subseteq p$
$p \wedge (p \mathrel{\Box\!\!\rightarrow} q) \to q$	$p \cap \Box_{R_p}(q) \subseteq q$	$R_p[x] \supseteq p$
$(p \mathrel{\Box\!\!\rightarrow} q) \to (p \wedge r \mathrel{\Box\!\!\rightarrow} q)$	$\Box_{R_p}(q) \subseteq \Box_{R_{p\cap q}}(q)$	$R_p[x] := \bigcup_{p \supseteq U} R_U[x]$
$(p \mathrel{\Box\!\!\rightarrow} q) \wedge (q \mathrel{\Box\!\!\rightarrow} r) \to p \mathrel{\Box\!\!\rightarrow} r$	$\Box_{R_p}(q) \cap \Box_{R_q}(r) \subseteq \Box_{R_p}(r)$	$R_p[x] := \bigcup\{R_U[x] \mid R_U[x] \subseteq p\}$
$(p \mathrel{\Box\!\!\rightarrow} q) \vee (p \mathrel{\Box\!\!\rightarrow} \neg q)$	$\Box_{R_p}(q) \cup \Box_{R_p}(X \setminus \downarrow q) = X$	The empty relation

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A mixed logic with binary operators*

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1 Introduction

Betweenness relations—well-known from geometry—are probably the most deeply studied ternary relations in logic and mathematics. As the class of betweenness relations is not modally definable, to investigate algebraic properties of betweenness, in (Düntsch et al., 2023) we worked with the so-called *PS-algebras*, i.e., expansions of the standard (binary) modal possibility algebras $\langle A, f\rangle$ with a binary *sufficiency* operator $g: A \times A \to A$ satisfying the following two conditions:

(i) if $x = \mathbf{0}$ or $y = \mathbf{0}$, then $g(x, y) = \mathbf{1}$ (co-normality),

(ii) $g(x, y) \cdot g(x, z) = g(x, y + z)$ and $g(y, x) \cdot g(z, x) = g(y + z, x)$ (co-additivity).

From the outset, there is no connection between the operators f and g. Thus, any meaningful interplay between them must be forced explicitly. One of the possible solutions is expanding the axioms with the following first-order condition

$$a \neq \mathbf{0} \text{ and } b \neq \mathbf{0} \to g(a, b) \leq f(a, b). \tag{wMIA}$$

PS-algebras satisfying (wMIA) are called *weak mixed algebras* or just *weak MIAs*. In (Düntsch et al., 2023), we showed that binary weak MIAs—counterparts of the unary wMIAs introduced by Düntsch et al. (2017)—can algebraically express certain axioms of betweenness relations.

2 The binary logic $K^{\#}$

The algebras from (Düntsch et al., 2017) have the following interesting property: the elements of the equational class generated by the weak MIAs are the algebraic models of the logic $K^{\sim}$, presented by Gargov et al. (1987). Our first objective is to develop the logic $K^{\#}$ (the counterpart of $K^{\sim}$) for the binary case using the copying construction of Vakarelov (1989) adapted for our needs. This is a Boolean logic with a set Var of propositional variables, a constant $\top$, and two extra binary modalities $\Box$ and $\boxminus$ with duals $\Diamond$ and $\diamondminus$. A *ternary frame* is a structure $\mathfrak{F} := \langle W, R, S\rangle$ where W is a nonempty set, and R, S are ternary relations on W. $\mathfrak{F}$ is called a *weak MIA frame*, if $S \subseteq R$. In (Düntsch et al., 2023), it was proved that the complex algebra of a weak MIA frame is a weak MIA and that the canonical frame of a weak MIA is a weak MIA frame. The class of weak MIA frames is decisive in the determination of the relational models of the logic $K^{\#}$. Indeed, we will prove the following theorems:

Theorem 2.1. *$K^{\#}$ is sound and complete with respect to wMIA frames.*

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Models are structures $\mathfrak{M} \coloneqq \langle W, R, S, v \rangle$ where $\langle W, R, S \rangle$ is a weak MIA frame and $v \colon \mathsf{Var} \to 2^W$ is a valuation. A model $\mathfrak{M}$ is called *special*, if $R = S$.

Theorem 2.2. *If $\mathfrak{M} \coloneqq \langle W, R, S, v \rangle$ is a model of $K^{\#}$, then there is a special model $\underline{\mathfrak{M}} \coloneqq \langle \underline{W}, \underline{R}, \underline{v} \rangle$ such that $\mathfrak{M}$ and $\underline{\mathfrak{M}}$ are modally equivalent.*

3 The class Eq(wMIA)

Let **wMIA** be the class of (binary) weak MIAs. Our second objective is to exhibit an axiom system for the equational class $\mathbf{V}$ of algebraic models of $K^{\#}$ and to prove that $\mathbf{V} = \mathbf{Eq}(\mathbf{wMIA})$, i.e., $\mathbf{V}$ is the variety generated by **wMIA**.

In the case of unary modalities investigated in (Düntsch et al., 2017) a unary PS-algebra $\langle A, f, g \rangle$ is a weak MIA if and only if the mapping defined by $u'(a) \coloneqq f^{\partial}(a) \cdot g(-a)$ is the dual of the unary discriminator. We have shown in (Düntsch et al., 2023) that for binary modalities such equivalence does not hold any more, and the weaker condition (**di**)

$$(\forall a, b \in A)[a \cdot b \neq \mathbf{0} \to g(a, b) \leq f(a, b)]. \tag{\textbf{di}}$$

is necessary and sufficient for the discriminator to exist. This observation leads us to a definition of the class of dMIAs (denoted by **dMIA**) as composed of PS-algebras that satisfies (**di**). We will exhibit an axiom system for the variety generated by **dMIA** and we will show that it is a proper subvariety of $\mathbf{Eq}(\mathbf{wMIA})$.

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The preserving non-falsity companion of the Nilpotent Minimum Logic

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We recall that a logic L is said to be *paraconsistent* with respect to a negation connective $\neg$ when it contains a $\neg$-contradictory but not trivial theory. Assuming that L is (at least) Tarskian, this is equivalent to say that the $\neg$-explosion rule $\frac{\varphi \quad \neg\varphi}{\psi}$ is not valid in L.

The 3-valued logic J_3 introduced by D'Ottaviano and da Costa in [2] is one of the well known paraconsistent logics and it can be defined (up to language) as the logic given by the matrix $\langle \mathbf{MV_3}, \{\frac{1}{2}, 1\}\rangle$ where $\mathbf{MV_3}$ is the 3 element MV-chain. Notice that J_3 is strongly related with the 3-valued Łukasiewicz logic Ł$_3$ as $\langle \mathbf{MV_3}, \{1\}\rangle$ is a matrix semantics for Ł$_3$. Moreover, these two logics are equivalent deductive systems in the Blok-Pigozzy sense [1]. Notice that, while Ł$_3$ is explosive and truth-preserving (1 being full truth), J_3 is paraconsistent and non-falsity-preserving, because it preserves every element different from 0 (0 being false). We call J_3 the *non-falsity companion of* Ł$_3$.

The *nilpotent minimum logic*, NML for short, was firstly introduced by Esteva and Godo in [3] in order to formalize the logic of the nilpotent minimum t-norm, that was defined by Fodor in [4] as an example of an involutive left continuous t-norm which is not continuous. NML is obtained from the monoidal t-norm logic MTL defined in [3], by adding the involutive condition axiom (INV) $\neg\neg\varphi \to \varphi$ and the (weak) nilpotent minimum condition axiom (WNM) $(\psi * \varphi \to \bot) \vee (\psi \wedge \varphi \to \psi * \varphi)$. It is well known that NML is algebraizable and the class $\mathbb{NM}$ of all nilpotent minimum algebras is its equivalent algebraic quasivariety semantics [3]. Moreover, NML is sound and strong complete with respect the standard NM-algebra $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ [7]. That is, NML is the logic defined by the matrix $\langle [\mathbf{0},\mathbf{1}]_{\mathbf{NM}}, \{1\}\rangle$. The aim of this talk is to axiomatize and characterize the non-falsity companions of NML and its axiomatic extensions.

Let $\mathbf{A}$ be a subalgebra of $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$, then the finitary logic L defined by $\langle \mathbf{A}, \{1\}\rangle$ is an axiomatic extension (not necessarily proper) of NML. We call nf-L the non-falsity companion of L. That is, nf-L is the finitary logic defined by the matrix $\langle \mathbf{A}, (0,1] \cap A\rangle$. Consider now the following *restricted* inference rule, which is intended for axiomatising nf-L::

- Restricted Square Modus Ponens for L (r-MP^2 for L):

  From φ and $\varphi \to \neg(\neg\psi)^2$ derive ψ, whenever $\vdash_L \varphi \to \neg(\neg\psi)^2$.

It is not hard to see that from (r-MP^2 for L) we can derive the following restricted version of Modus Ponens:

- Restricted Modus Ponens for L (r-MP for L):

  From φ and $\varphi \to \psi$ derive ψ, whenever $\vdash_L \varphi \to \psi$

Note that both inference rules involve conditions on the derivability of formulas in the logic L. Since any axiomatic extension of NML is complete w.r.t at most two subalgebras of $[\mathbf{0},\mathbf{1}]_{\mathbf{NM}}$ [5] we obtain the following result.

Theorem 1. *Let L be an axiomatic extension of NML. The following axiomatization*

- *Axioms: those of L*
- *Rules: Adjunction* $\dfrac{\varphi \quad \psi}{\varphi \wedge \psi}$ *and* *(r-MP^2) for L*

is a sound and complete axiomatisation of nf*-L.*

For the case of finite-valued axiomatic extensions NM_n, unlike the Lukasievicz case [1, Th.5.2], we prove that nf-NM_n is not equivalent to NM_n. With an abuse of language, $\mathcal{N}_k$ denotes the matrix $\langle \mathbf{NM_k}, \{1\}\rangle$ and $\mathcal{J}_k$ will denote the matrix $\langle \mathbf{NM_k}, \{\frac{1}{k-1}, \frac{2}{k-1}, \ldots, 1\}\rangle$ where $\mathbf{NM_k}$ is the k-element NM-chain. It is shown in [6] that any finitary extension of NM_n is complete w.r.t. following set of matrices $\{\mathcal{N}_{2k}, \mathcal{N}_{2m+1}, \mathcal{N}_2 \times \mathcal{N}_{2r+1}\}$ for some $0 \leqslant m \leqslant r \leqslant k \leqslant n$, For the case of nf-NM_n we cannot accomplish this reduction, but the following one that is restricted to finitary extensions defined by finite products of $\mathcal{J}_k$'s.

Theorem 2. *Let L be a finitary extension of* nf*-NML defined by $\mathcal{J}_{k_1} \times \cdots \times \mathcal{J}_{k_s}$. Then L is complete w.r.t a finite set of the following matrices:*

(i) *$\mathcal{J}_n$ for some positive integer $n > 1$.*

(ii) *$\mathcal{J}_n \times \mathcal{J}_k$ for some positive integers $n \neq k$.*

(iii) *$\mathcal{J}_{2n} \times \mathcal{J}_{2k} \times \mathcal{J}_{2l+1}$ for some positive integers $l < n < k$.*

(iv) *$\mathcal{J}_{2n} \times \mathcal{J}_{2m+1} \times \mathcal{J}_{2l+1}$ for some positive integers $m < n$ and $m < l$.*

Moreover every different matrix of these four types defines a different logic

Finally, next result charcaterizes all finite maximal paraconsistent extensions nf-NML

Theorem 3. *The only finite matrices defining maximal paraconsitent extesnions of* nf*-NML are $\mathcal{J}_3$, $\mathcal{J}_4$ and $\mathcal{J}_3 \times \mathcal{J}_4$.*

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On Monadic De Morgan Monoids*

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Monadic Boolean algebras were first systematically studied by Halmos [3] and generalized to Polyadic Boolean algebra [2] which serves as a framework for algebrization of predicate logic. One of the important theorems is that that every monadic Boolean algebra is a subdirect union of functional monadic Boolean algebras. On the other hands, the structure of De Morgan monoids (DMM) have been extensively studied in [4, 5] and its connection with relevance logic [1]. In [6], Andrew Tedder proposed an algebraic framework for Mares-Goldblatt semantics for quantified relevance logics by using as an example De Morgan monoid with a Mares-Goldblatt style interpretation of the quantifiers to study quantified relevance logic. This sheds a light on generalization of monadic De Morgan monoids and polyadic De Morgan monoids as an algebraization of quantified relevance logic. In this talk, we will report our work in progress in this direction of study.

We start with the definition of functional monadic DMM.

Definition 1 (Functional Monadic De Morgan Monoid)**.** *If $\mathfrak{B}$ is a DMM, X is a non-empty set, then the structure $\langle \mathfrak{A}; \wedge, \vee, \sim, \circ, \rightarrow, 1\rangle$ is a $\mathfrak{B}$-valued functional monadic DMM, if the following conditions hold :*

1. $\mathfrak{A} \subseteq \mathfrak{B}^X$;
2. *$\mathfrak{A}$ is closed under the 'lifted' operations $\wedge, \vee, \sim, \circ, \rightarrow$, and contains 1, where*
   - (a) $1(x) =^{\mathfrak{B}} 1$, *for all* $x \in X$;
   - (b) $(\sim p)(x) =^{\mathfrak{B}} \sim (p(x))$, *for* $p \in \mathfrak{A}$;
   - (c) $(p \otimes q)(x) =^{\mathfrak{B}} p(x) \otimes q(x)$, *for* $p, q \in \mathfrak{A}$ *and* $\otimes \in \{\wedge, \vee, \sim, \circ, \rightarrow\}$
3. *The constant function $\forall p$ exists in $\mathfrak{A}$, for each $p \in \mathfrak{A}$, and hence the appropriate generalized meets and joins exists in $\mathfrak{B}$, where we define:*

$$R(p) =_{df} \{p(x) : x \in X\}$$

$$\forall p(x) =_{df} \bigwedge^{\mathfrak{B}} R(p)$$

Then we demonstrate the following property of functional monadic DMM.

Lemma 1. *A functional universal quantifier $\forall$ on a functional monadic DMM satisfies the following:*

(Q_1) $\forall 1 = 1$

(Q_2) $\forall p \leq p$

(Q_3) $\forall (p \wedge q) = \forall p \wedge \forall q$

(Q_4) $\forall\forall p = \forall p = \neg\forall\neg\forall p$

(Q_5) $\forall (p \rightarrow q) \leq (\forall p \rightarrow \forall q)$

(Q_6) $\forall (\forall p \rightarrow \forall q) = \forall p \rightarrow \forall q$

(Q_7) $\forall (p \vee q) \leq \neg\forall\neg p \vee \forall q$

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On the other hand, we can define a universal quantifier to be a mapping satisfying certain conditions as follows :

Definition 2 (Universal Quantifier)**.** *$\mathfrak{A}$ be an DMM. A* (Universal) Quantifier *is a map $\forall : \mathfrak{A} \to \mathfrak{A}$ that satisfies (Q_1)–(Q_7) (defined in Lemma 1).*

Some important facts about universal quantifiers can be derived :

Lemma 2. *Let $\mathfrak{A}$ be an DMM and $\forall$ is a quantifier on $\mathfrak{A}$. The following properties hold:*

1. $\forall 0 = 0$
2. $p \in \forall(\mathfrak{A})$ *iff* $\forall p = p$.
3. *If* $\forall p \leq q$ *then* $\forall p \leq \forall q$.
4. *If* $p \leq q$ *then* $\forall p \leq \forall q$.
5. $\forall(p \to q) \leq (\neg\forall\neg p \to \neg\forall\neg q)$
6. $p \leq \forall\neg\forall\neg p$
7. $\forall(\forall p \to q) \leq (\forall p \to \forall q)$

Definition 3. *A Monadic DMM is a tuple $\langle \mathfrak{A}, \forall \rangle$ where $\mathfrak{A}$ is an DMM, and $\forall$ is a quantifier on $\mathfrak{A}$.*

In the end of this talk, we will briefly address the problem of two Representation theorems in monadic De Morgan monoid either in terms of functional monadic DMM or in terms of subdirectly irreducible monadic DMM.

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Degrees of incompleteness of implicative logics: the trichotomy theorem

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The degree of incompleteness of a logic L is the cardinality of the set of logics with the same Kripke frames as L [3]. Blok's celebrated *dichotomy theorem* states that every normal modal logic has degree of incompleteness either 1 or $2^{\aleph_0}$ [1]. The aim of this talk is to prove a *trichotomy theorem* for the degrees of incompleteness of the *implicative logics*, i.e., the axiomatic extensions of the implicative fragment $\mathsf{IPC}_{\to}$ of the intuitionistic logic IPC. In particular, we will prove that the degree of incompleteness of any implicative logic is 1, $\aleph_0$, or $2^{\aleph_0}$. Notably, the degree of incompleteness of an implicative logic coincides with its *degree of the finite model property*, as defined in [4].

In what follows, we make the above statement precise. A formula of the *intuitionistic propositional calculus* IPC is said to be *implicative* when it contains no connective other than $\to$.

Definition 1. The *implicative fragment* of IPC is the set

$$\mathsf{IPC}_{\to} := \{\varphi \in \mathsf{IPC} : \varphi \text{ is an implicative formula}\}.$$

Notably, $\mathsf{IPC}_{\to}$ coincides with the set of implicative formulas φ such that $\mathsf{Hil} \vDash \varphi$, where Hil is the variety of *Hilbert algebras*, i.e., the class of subalgebras of the *implicative reducts* $\langle A; \to\rangle$ of Heyting algebras [2]. Since $x \to x$ is a constant term in every Hilbert algebra, we will use the shorthand $1 := x \to x$.

Definition 2. An *implicative logic* is a set of implicative formulas containing $\mathsf{IPC}_{\to}$ that, moreover, is closed under modus ponens and uniform substitutions.

When ordered under the inclusion relation, the set of implicative logics forms a complete lattice $\mathrm{Ext}(\mathsf{IPC}_{\to})$ which is dually isomorphic to the lattice $\Lambda(\mathsf{Hil})$ of varieties of Hilbert algebras. This dual isomorphism is witnessed by the maps $\mathsf{Var}(-)$ and $\mathsf{Log}(-)$ defined for every $\mathsf{L} \in \mathrm{Ext}(\mathsf{IPC}_{\to})$ and $\mathsf{V} \in \Lambda(\mathsf{Hil})$ as

$$\begin{aligned}\mathsf{Var}(\mathsf{L}) &:= \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \vDash \mathsf{L}\};\\ \mathsf{Log}(\mathsf{V}) &:= \{\varphi : \varphi \text{ is an implicative formula such that } \mathsf{V} \vDash \varphi\}.\end{aligned}$$

Given a poset $\boldsymbol{X}$ and a set Γ of implicative formulas we write $X \Vdash \Gamma$ when Γ is valid in $\boldsymbol{X}$, viewed as an intuitionistic Kripke frame.

Definition 3. The *span* of an implicative logic L is the set

$$\mathsf{span}(\mathsf{L}) := \{\mathsf{L}' \in \mathrm{Ext}(\mathsf{IPC}_{\to}) : \boldsymbol{X} \Vdash \mathsf{L} \text{ iff } \boldsymbol{X} \Vdash \mathsf{L}', \text{ for every poset } \boldsymbol{X}\}.$$

Furthermore, the *degree of incompleteness* of L is $\mathsf{deg}(\mathsf{L}) := |\mathsf{span}(\mathsf{L})|$.

Before stating our main result characterising the degree of incompleteness of the implicative logics, we need to introduce two classes of varieties. To this end, recall that a subset F of a Hilbert algebra $\boldsymbol{A}$ is an *implicative filter* if it contains 1 and for every $\{a, b\} \subseteq A$, if $\{a, a \to b\} \subseteq F$, then $b \in F$.

Definition 4. Given $n \in \mathbb{N}$, we say that a Hilbert algebra $\boldsymbol{A}$ has *depth* $\leqslant n$ when the poset of its meet irreducible implicative filters does not contain $(n+1)$-elements chains. Then, the following is a variety:

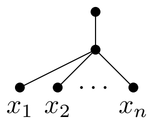
$$\mathsf{D}_n \coloneqq \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \text{ has depth} \leqslant n\}.$$

In order to define the second class of varieties, with every poset $\boldsymbol{X} = \langle X; \leqslant \rangle$ with maximum $\top$ we associate a binary operation $\to$ on X defined by the rule

$$x \to y \coloneqq \begin{cases} \top & \text{if } x \leqslant y; \\ y & \text{otherwise.} \end{cases}$$

Then, $\mathsf{H}(\boldsymbol{X}) \coloneqq \langle X; \to \rangle$ is a Hilbert algebra with underlying partial order $\leqslant$. Lastly, we denote the smallest variety containing a class of algebras K by $\mathbb{V}(\mathsf{K})$.

Definition 5. For each $n \in \mathbb{Z}^+$ let $\boldsymbol{B}_n \coloneqq \mathsf{H}(\mathbb{B}_n)$, where $\mathbb{B}_n$ is the poset depicted below:



Furthermore, let

$$\mathsf{B}_n \coloneqq \mathbb{V}(\boldsymbol{B}_n) \text{ and } \mathsf{B}_\omega \coloneqq \mathbb{V}(\{\boldsymbol{B}_n : n \in \mathbb{Z}^+\}).$$

Our main result takes the following form:

Trichotomy Theorem. *The following conditions hold for an implicative logic* L*:*

(i) $\mathsf{deg}(\mathsf{L}) = 1$ *if and only if* $\mathsf{L} = \mathsf{IPC}_\to$ *or* $\mathsf{L} = \mathsf{Log}(\mathsf{D}_n)$ *for some* $n \in \mathbb{N}$*;*

(ii) $\mathsf{deg}(\mathsf{L}) = \aleph_0$ *if and only if* $\mathsf{L} = \mathsf{Log}(\mathsf{B}_\omega)$ *or* $\mathsf{L} = \mathsf{Log}(\mathsf{B}_n)$ *for some* $n \in \mathbb{Z}^+$*;*

(iii) $\mathsf{deg}(\mathsf{L}) = 2^{\aleph_0}$ *otherwise.*

We remark that the problem of determining which are the degrees of incompleteness of intermediate logics is an outstanding open problem and hope that this talk will stimulate research in this direction.

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Amalgamation in Varieties of BL-algebras

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BL-algebras are the equivalent algebraic semantics of Hájek's basic fuzzy logic **BL**. The latter was shown in [7] to be the logic of continuous t-norms, and **BL** and BL-algebras have subsequently attracted a great deal of attention from both fuzzy and substructural logicians. In the last twenty-five years, an extensive body of work on **BL** and BL-algebras has developed and the literature now offers a rather mature theory (see, e.g., [1, 2, 5, 6, 9] for a sample). One outstanding problem in this theory, however, is to completely classify the varieties of BL-algebras that have the amalgamation property—or, in an equivalent logical formulation, to completely classify the axiomatic extensions of **BL** that have the deductive interpolation property.

This problem has proven quite challenging. In [12], Montagna showed that many of the most natural varieties of BL-algebras (including the variety of all BL-algebras) have the amalgamation property, but that there are uncountably many varieties of BL-algebras that do not. Later, by working with some technical hypotheses on the form of generating algebras for the varieties in question, Aguzzoli and Bianchi provided a partial classification of varieties of BL-algebras with the amalgamation property [3]. They later sharpened this classification in [4], but their results still stopped short of an exhaustive classification.

In this work, we provide just such an exhaustive classification of varieties of BL-algebras that have the amalgamation property, consequently giving a complete classification of the axiomatic extensions of Hájek's basic logic that have the deductive interpolation property. In particular, we show that there are just countably many of these, answering the question posed by Montagna in [12, Section 7].

Our classification proceeds in three steps. First, we obtain a new equivalent formulation of the amalgamation property that is better suited to studying amalgamation in many varieties generated by linearly ordered algebras, including varieties of BL-algebras. We say that an extension $\mathbf{A} \leq \mathbf{B}$ is *essential* provided that $\theta \neq \Delta_B$ implies $\theta \cap A^2 \neq \Delta_A$, and we say that an embedding $\phi\colon \mathbf{A} \to \mathbf{B}$ is *essential* whenever $\phi[\mathbf{A}] \leq \mathbf{B}$ is. A span $\langle i_1 \colon \mathbf{A} \to \mathbf{B}, i_2 \colon \mathbf{A} \to \mathbf{C}\rangle$ of algebras is *essential* provided that i_2 is an essential embedding, and a class of algebras K has the *essential amalgamation property* if for any essential span $\langle i_1 \colon \mathbf{A} \to \mathbf{B}, i_2 \colon \mathbf{A} \to \mathbf{C}\rangle$ in K, there exists $\mathbf{D} \in \mathsf{K}$ and embeddings $j_1 \colon \mathbf{B} \to \mathbf{D}$ and $j_2 \colon \mathbf{C} \to \mathbf{D}$ such that $j_1 \circ i_1 = j_2 \circ i_2$.

Theorem 1. *Let V be a variety and $\mathsf{V}_{\mathrm{FSI}}$ be the class of finitely subdirectly irreducible members of V. Suppose that V has the congruence extension property and that $\mathsf{V}_{\mathrm{FSI}}$ is closed under subalgebras and homomorphic images. Then V has the amalgamation property if and only if $\mathsf{V}_{\mathrm{FSI}}$ has the essential amalgamation property.*

In the second step toward our classification, we apply the previous theorem to study amalgamation for 0-free subreducts of BL-algebras, often called *basic hoops*. The finitely subdirectly irreducible basic hoops are precisely the totally ordered ones, so Theorem 1 provides a powerful criterion for the amalgamation property in this context. Together with the well-known decomposition of totally ordered BL-algebras as ordinal sums of Wajsberg hoops and the classification of varieties of Wajsberg hoops with the amalgamation property [11, Theorem 63], we use Theorem 1 to give a tangible description of the poset of varieties of basic hoops with the amalgamation property. In particular, we show that this poset can be partitioned into a countably infinite family of finite intervals and give concrete descriptions of the latter. Thus:

Theorem 2. *There are only countably many varieties of basic hoops that have the amalgamation property.*

In the third step toward our classification, we use Theorem 2 along with the well-known classification of varieties of MV-algebras with the amalgamation property (see [8]) to describe all varieties of BL-algebras with the amalgamation property. Like for basic hoops, these turn out to all fall into one of countably infinitely many finite intervals, which we may concretely describe. Thus:

Theorem 3. *There are only countably many varieties of BL-algebras that have the amalgamation property.*

By applying the well-known connection between the amalgamation property and the deductive interpolation property for algebraizable logics, we may deduce the following result from Theorems 2 and 3.

Theorem 4. *There are only countably many axiomatic extensions of Hájek's basic fuzzy logic that have the deductive interpolation property. The same holds for axiomatic extensions of the negation-free fragment of Hájek's basic fuzzy logic.*

More information can be found in our preprint [10].

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Interpolation in Some Modal Substructural Logics

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The *deductive interpolation property* stipulates that, for the consequence relation $\vdash$ of some propositional logic and any set of formulas $\Gamma \cup \{\phi\}$,

> if $\Gamma \vdash \phi$, then there exists a set of formulas Γ' whose variables are among those contained in both Γ and ϕ such that $\Gamma \vdash \Gamma'$ and $\Gamma' \vdash \phi$.

Deductive interpolation has been studied in a range of different contexts. Famously, Maksimova showed in [7] that, of the continuum-many consistent superintuitionistic logics, only 7 have the deductive interpolation property (equivalent in that context to the better known Craig interpolation property). Later, in [8], she showed that there are at most 49 consistent normal extensions of the modal logic **S4** with the deductive interpolation property.

In this work, we study deductive interpolation in a substructural environment that combines these. In particular, we examine expansions of substructural logics with the exchange rule—but possibly lacking the contraction or weakening rules—by **S4**-like modalities. Our work proceeds by first exhibiting continuum-many axiomatic extensions of the full Lambek calculus with exchange $\mathbf{FL_e}$ that have the deductive interpolation property, and then showing that—because of the special form of the extensions constructed—each of these may be expanded by an **S4**-like modality. In this fashion, we also obtain continuum-many **S4**-like modal expansions of $\mathbf{FL_e}$ that have deductive interpolation. Previously, only countably many substructural logics were known to have the deductive interpolation property, so our work contributes to the general theory of interpolation in substructural logics as well as stands in contrast to Maksimova's results.

For suitable algebraizable logics, there is a well-known connection between the deductive interpolation property for a logic and the amalgamation property for its associated class of algebraic models (see, e.g., [2]). Consequently, much like Maksimova, our work focuses on the study of amalgamation in appropriately chosen algebraic models. The fundamental algebraic structures we consider are *FL_e-algebras*, i.e., algebras of the form $\langle A, \wedge, \vee, \cdot, \to, 0, 1\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a commutative monoid, and $x \cdot y \leq z$ if and only if $x \leq y \to z$. To treat modalities, we define an *$S4FL_e$-algebra* to be an expansion of an FL_e-algebra by an additional unary operation $\Box$ that satisfies the identities

1. $\Box(x \wedge y) = \Box x \cdot \Box y$.
2. $\Box\Box x = \Box x \leq x$.
3. $\Box 1 = 1$.

To find continuum-many logics with deductive interpolation, we construct continuum-many varieties of FL_e-algebras with the amalgamation property (see [3] for relevant definitions). These varieties are constructed by first considering suitably chosen quasivarieties of abelian groups, each with the amalgamation property. The abelian groups contained in these quasivarieties are then transformed into FL_e-algebras using a construction that preserves the amalgamation property, and the latter are used as generating algebras for the varieties we are interested in. The examples we construct are sufficiently transparent to lift the amalgamation property from the generating algebras using existing tools (see, e.g., [3, 9]), but also sufficiently flexible that they may be expanded into $S4FL_e$-algebras so as to keep the amalgamation property. Thus:

Theorem 1.

1. *There are continuum-many varieties of FL_e-algebras that have the amalgamation property*
2. *There are continuum-many varieties of $S4FL_e$-algebras that have the amalgamation property.*

FL_e-algebras are well-known to algebraize the consequence relation of the full Lambek equipped with the exchange rule [5]. Likewise, $S4FL_e$-algebras algebraize an **S4**-like modal variant of the full Lambek calculus with exchange, which we call $\mathbf{S4FL_e}$. The logics algebraized by the varieties considered in Theorem 1 all enjoy local deduction theorems, so by well-known bridge theorems linking amalgamation and deductive interpolation we obtain the following:

Theorem 2.

1. *There are continuum-many axiomatic extensions of* $\mathbf{FL_e}$ *that have the deductive interpolation property.*
2. *There are continuum-many axiomatic extensions of* $\mathbf{S4FL_e}$ *that have the deductive interpolation property.*

The techniques we use to obtain Theorems 1 and 2 are extremely flexible, and also allow us to obtain a host of similar results for related logics. Of these, we mention only Girard's celebrated linear logic [6], which is algebraized by certain expansions of $S4FL_e$-algebras (see [1]):

Theorem 3. *Classical linear logic has continuum-many axiomatic extensions with the deductive interpolation property.*

Further information on this work may be found in our preprint [4].

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Temporal Logic of a Sequence of Finite Linear Processes

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We provide a sound and complete axiomatization of a temporal logic of a sequence of finitely many finite linear structures linked by surjective bounded morphisms.

Finite linear structures, *i.e.*, finite sets with a strict linear ordering, naturally arise as representations of a discrete, bounded time flow. Many domains of our everyday practice including time series [1], scene analysis [6], chain-of-responsibility design pattern in programming [3], [5], *etc.* involve a finite linear structure to represent a sequence of consecutive steps. A familiar example of such a structure is a movie represented as a sequence of individual frames.

In such scenarios, it is often natural to group consecutive elements into conceptually meaningful units in such a way that these units inherit the temporal order of the original structure. Moreover, this process can be repeated finitely many times. For a typical example of what is meant here consider a set of movie frames, grouped into episodes, these further grouped into scenes, which finally form acts. The structure of episodes inherits the temporal order from the ordering of individual frames. The same is true for the structure of scenes, and that of acts.

Definition 1. *A **TES** (Temporal Event Structure) is* $(F_1, \ldots, F_n, <_1, \ldots, <_n, f_1, \ldots, f_{n-1})$ *where* $(F_i, <_i)$ *are finite strict linear orders, while* $f_i : F_i \twoheadrightarrow F_{i+1}$ *are onto monotone maps, where monotone means* $f_i(a) \leq_{i+1} f_i(b)$ *for all* $a \leq_i b$*. Let* $F \coloneqq \bigcup_{i=1}^{n} F_i$, $< \coloneqq \bigcup_{i=1}^{n} <_i$, $f \coloneqq \bigcup_{i=1}^{n-1} f_i$.

The language $\mathcal{L}$ is given by: $\phi = \ p \mid \neg\phi \mid \phi \wedge \phi \mid \Box_{\downarrow}\phi \mid \Box_{\uparrow}\phi \mid \Box_{\rightarrow}\phi \mid \Box_{\leftarrow}\phi$, where p ranges over proposition symbols. Other logical symbols are defined as usual.

Our intention is to interpret the language $\mathcal{L}$ over an arbitrary **TES** $(F, <, f)$ in such a way that $\Box_{\rightarrow}$ and $\Box_{\leftarrow}$ range over $(F, <, >)$ while $\Box_{\downarrow}$ and $\Box_{\uparrow}$ range over (F, f, f^{-1}). Denote the class of all **TES** with n fixed by $\mathbf{T_n}$. The logic $\mathsf{Log}(\mathbf{T_n})$ is the set of all formulas of $\mathcal{L}$ valid on all structures in $\mathbf{T_n}$. $\mathsf{Log}(\mathbf{T_2})$ was investigated in a recent paper [2].

In this contribution we present an axiomatization of $\mathsf{Log}(\mathbf{T_n})$ for arbitrary fixed $n > 1$. Let $\mathbf{L_n}$ be the least subset of $\mathcal{L}$ containing the following set of axioms and closed under the standard rules of uniform substitution, modus ponens and necessitation.

- All classical tautologies, standard axioms of modal logic **K** for each modal operator;

`Inv`: $p \to \Box_{\leftarrow}\Diamond_{\rightarrow} p \wedge \Box_{\rightarrow}\Diamond_{\leftarrow} p$ $p \to \Box_{\uparrow}\Diamond_{\downarrow} p \wedge \Box_{\downarrow}\Diamond_{\uparrow} p$	`GL`: $\Box_{\rightarrow}(\Box_{\rightarrow} p \to p) \to \Box_{\rightarrow} p$ $\Box_{\leftarrow}(\Box_{\leftarrow} p \to p) \to \Box_{\leftarrow} p$	`NoBranching`: $\Diamond_{\rightarrow}\Diamond_{\leftarrow} p \to \Diamond_{\rightarrow} p \vee p \vee \Diamond_{\leftarrow} p$ $\Diamond_{\leftarrow}\Diamond_{\rightarrow} p \to \Diamond_{\rightarrow} p \vee p \vee \Diamond_{\leftarrow} p$
`Level`: $\bigwedge_{k=1}^{n-1} \left(\Box_{\uparrow}^{k} \bot \to \Diamond_{\downarrow}^{n-k} \top\right)$	`Length`: $\Box_{\downarrow}^{n} \bot$	`Coherence`: $\bigwedge_{k=1}^{n-1} \left(\Diamond_{\downarrow}^{k} \top \to \Box_{\rightarrow}\Diamond_{\downarrow}^{k} \top \wedge \Box_{\leftarrow}\Diamond_{\downarrow}^{k} \top\right)$
`Surj`: $\bigvee_{k=1}^{n-1} \left(\Diamond_{\uparrow}^{k} \top \to \Box_{\rightarrow}\Diamond_{\uparrow}^{k} \top \wedge \Box_{\leftarrow}\Diamond_{\uparrow}^{k} \top\right)$	`Bounded`: $\Diamond_{\downarrow}\Diamond_{\rightarrow} p \to \Diamond_{\rightarrow}\Diamond_{\downarrow} p$	`DomConn`: $\Diamond_{\downarrow}\Diamond_{\uparrow} p \to \Diamond_{\rightarrow} p \vee p \vee \Diamond_{\leftarrow} p$
`Func`: $p \to \Box_{\uparrow}\Box_{\downarrow} p$	`Monot`: $\Diamond_{\rightarrow}\Diamond_{\downarrow} p \to \Box_{\downarrow}(p \vee \Diamond_{\rightarrow} p)$	

Abstract Kripke semantics for $\mathcal{L}$ is provided by Kripke frames $\mathcal{F} = (W, R, R_f, R', R'_f)$ where W is a nonempty set and each of $R, R_f, R', R'_f \subseteq W \times W$ is a binary relation.

Definition 2. *We will say that a frame $\mathcal{F} = (W, R, R_f, R', R'_f)$ is an $\mathbf{L_n}$-frame if the following conditions are satisfied: $W = \bigcup_{i=1}^{n} W_i$ where for all distinct $i, j \leq n$ we have $W_i \neq \emptyset$ and $W_i \cap W_j = \emptyset$; $R' = R^{-1}$; R, R' are non-branching, transitive and well-founded and $R = \bigcup_{i=1}^{n} R_i$ where $R_i = R \cap (W_i \times W_i)$ for $i \leq n$; $R_f \cap (W_i \times W_{i+1})$ is a surjective bounded morphism with respect to R_i and R_{i+1}; $R'_f = R_f^{-1}$ and R_f is domain connected [2, Def. 3.6].*

Theorem 3. *For an arbitrary frame $\mathcal{F}$ it holds that $\mathcal{F} \models \mathbf{L_n}$ iff $\mathcal{F}$ is an $\mathbf{L_n}$-frame.*

Clearly a disjoint union of $\mathbf{L_n}$-frames is again an $\mathbf{L_n}$-frame. This implies that $\mathbf{L_n}$-frames can be infinite, and fail the trichotomy property for R_i, $i \leq n$, while our intended models, **TES**s are finite with $<_i$ trichotomous. To retain finiteness and trichotomy, we focus our attention on connected $\mathbf{L_n}$-frames, *i.e.* on $\mathbf{L_n}$-frames which cannot be presented as a disjoint union of two $\mathbf{L_n}$-frames. It turns out that a connected $\mathbf{L_n}$-frame is in a way isomorphic to a **TES**.

Theorem 4. *In every connected $\mathbf{L_n}$-frame $\mathcal{F} = (W, R, R_f, R', R'_f)$ the set W is finite and each relation R_i is trichotomous.*

The class of connected $\mathbf{L_n}$-frames is modally undefinable since it is not closed under disjoint unions. The next theorem links connected $\mathbf{L_n}$-frames and **TES**s.

Theorem 5. *There is a one-to-one correspondence between the class $\mathbf{T_n}$ and the class of all connected $\mathbf{L_n}$-frames.*

The next theorem shows that each **TES** can be fully described by an $\mathcal{L}$-formula.

Theorem 6. *Given a* **TES** *$\mathcal{F} = (F, <, f)$ there is a formula $\phi_{\mathcal{F}} \in \mathcal{L}$ such that for an arbitrary* **TES** *$\mathcal{T}$ we have: $\mathcal{T} \models \phi_{\mathcal{F}}$ iff $\mathcal{T}$ is isomorphic to $\mathcal{F}$.*

Finally, we establish our main finding:

Theorem 7. *The logic $\mathbf{L_n}$ is sound and complete w.r.t. the class $\mathbf{T_n}$.*

It follows that the logic $\mathbf{L_n}$ has the finite model property and is decidable.

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Distributive lattice-ordered pregroups

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Abstract

We show that the variety of distributive ℓ-pregroups is generated by a single functional algebra, $\mathbf{F}(\mathbb{Z})$, and that it has a decidable equational theory. We also prove generation and decidability results for each of its n-periodic subvarieties.

1 Introduction

A *lattice-ordered pregroup* (*ℓ-pregroup*) is an algebra $(A, \wedge, \vee, \cdot, ^{\ell}, ^{r}, 1)$, where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, multiplication preserves the lattice order $\leq$, and for all x,

$$x^{\ell}x \leq 1 \leq xx^{\ell} \text{ and } xx^{r} \leq 1 \leq x^{r}x.$$

We often refer to x^{ℓ} and x^{r} as the *left* and *right inverse* of x, respectively. The well-studied lattice-ordered groups (ℓ-groups) are exactly the ℓ-pregroups where the two inverses coincide: $x^{\ell} = x^{r}$. Also, ℓ-pregroups constitute lattice-ordered versions of *pregroups*, which are ordered structures introduced by Lambek [11] in the study of applied linguistics, where they are used to describe sentence patterns in many natural languages; they have also been studied extensively by Buzkowski [1] and others in the context of mathematical linguistics in connection to context-free grammars. Pregroups where the order is discrete are exactly groups.

The main reason for our interest in ℓ-pregroups is that they are precisely the *involutive residuated lattices* that satisfy $x + y = xy$; in that respect their study is connected to the algebraic semantics of *substructural logics* [6].

It is easy to show that the underlying lattices of ℓ-groups are distributive. In [5] we show that ℓ-pregroups are semidistributive, but it remains an open problem whether every ℓ-pregroup is distributive. In this submission we focus on the variety DLP of *distributive ℓ-pregroups*.

In analogy to Cayley's theorem for groups, Holland's *embedding theorem* [9] shows that every ℓ-group can be embedded into a symmetric ℓ-group $\mathbf{Aut}(\mathbf{\Omega})$—the group of order-preserving permutations on a totally ordered set $\mathbf{\Omega}$. Also, Holland's *generation theorem* [10] states that $\mathbf{Aut}(\mathbb{Q})$ generates the variety of ℓ-groups and this is further used to show that the equational theory of ℓ-groups is decidable. In [2] we showed that every distributive ℓ-pregroup embeds into a functional ℓ-pregroup $\mathbf{F}(\mathbf{\Omega})$ (a generalization of a symmetric ℓ-group), where $\mathbf{\Omega}$ is a chain.

In this submission, which is based on [7], we improve this embedding theorem by showing that every distributive ℓ-pregroup embeds into $\mathbf{F}(\mathbf{\Omega})$, where $\mathbf{\Omega}$ is an ordinal sum of copies of the integers (we call such chains *integral*). This allows us to obtain an analogue of Holland's generation theorem: the ℓ-pregroup $\mathbf{F}(\mathbb{Z})$ generates the variety DLP. Furthermore, we use this result to prove the decidability of the equational theory of distributive ℓ-pregroups. The methods we use are based on the notion of *diagram*, which is a finitistic object that captures the failure of an equation. The diagrams situation in ℓ-pregroups is much more complex than in ℓ-groups, as one-sided inverses can pile up and computating them in a diagram is quite involved.

Time permitting, we will also discuss our work included in [8]. For every positive integer n, the functions f in $\mathbf{F}(\mathbb{Z})$ that are periodic and have period n end up being exactly the ones that satisfy $f^{\ell^n} = f^{r^n}$; in particular, the ones satifying $f^\ell = f^r$ are the order-preserving permutations on $\mathbb{Z}$. Taking this as inspiration, an element x in an ℓ-pregroup is called *n-periodic* if $x^{\ell^n} = x^{r^n}$; an ℓ-pregroup is called *n-periodic* if all of its elements are, and the corresponding variety is denoted by LP_n. In [3] we showed that $\mathsf{LP}_n \subseteq \mathsf{DLP}$, for all n. Using *n-periodic* diagrams we prove that the join of all of the LP_n's is exactly DLP; this is the analogue of the corresponding theorem for the variety of involutive residuated lattices that we proved in [4] using proof-theoretic methods.

Moreover, we get a representation theorem: every algebra in LP_n can be embedded in the subalgebra $\mathbf{F}_n(\mathbf{\Omega})$ of n-periodic elements of $\mathbf{F}(\mathbf{\Omega})$, for an integral chain $\mathbf{\Omega}$. We prove that DLP is also equal to the join of the varieties $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$, thus $\bigvee \mathsf{LP_n} = \bigvee \mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$, but unfortunately $\mathsf{LP_n} \neq \mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$ for every single n. By [10], $\mathsf{LP}_1 = \mathsf{V}(\mathbf{F}_1(\mathbb{Q}))$, but we show that $\mathsf{LP}_n \neq \mathsf{V}(\mathbf{F}_n(\mathbb{Q}))$, for all $n > 1$. In the end we find suitable chains $\mathbf{\Omega}_n$, such that $\mathsf{LP}_n = \mathsf{V}(\mathbf{F}_n(\mathbf{\Omega}_n))$, for every n; actually, we do better than that by identifying a single uniform chain: $\mathsf{LP}_n = \mathsf{V}(\mathbf{F}_n(\mathbb{Q} \overrightarrow{\times} \mathbb{Z}))$, for all n. This result is obtained by a deep analysis of the structure of n-periodic ℓ-pregroups. We prove that every such algebra can be embedded in a *wreath product* of an ℓ-group and $\mathbf{F}_n(\mathbb{Z})$, we analyze the global and local components and see how this is reflected on n-periodic *partition diagrams*.

We also prove that for every n, the equational theories of LP_n and of $\mathbf{F}_n(\mathbb{Z})$ are decidable, where the latter plays a crucial role for the former. The height (difference between input and output values) of a function in $\mathbf{F}_n(\mathbb{Z})$ involved in a failure of an equation needs to be controlled in order to obtain decidability. We show that functions in $\mathbf{F}_n(\mathbb{Z})$ decompose into translations and functions of short height. We use results from linear algebra to control the height of the automorphism part and compose this short piece back to obtain a new short function of $\mathbf{F}_n(\mathbb{Z})$.

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The Tree Structure of Conservative Commutative Residuated Lattices

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A residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is *commutative* and *idempotent* if its monoidal part $\langle A, \cdot, e\rangle$ is commutative and idempotent, that is, if it satisfies the equations $x \cdot y = y \cdot x$ and $x \cdot x \approx x$, respectively. And we call $\mathbf{A}$ a *residuated chain* if the order $\leqslant$ associated to its lattice part $\langle A, \wedge, \vee\rangle$ is total. In [1] we studied several classes of idempotent residuated chains and their generated varieties. In particular, we established a more symmetric version of Raftery's characterization theorem [2] for commutative idempotent residuated chains, obtaining also as a corollary (as in [2]) that they generate a locally finite variety.

In that work, it was instrumental the fact that the monoidal structure of any idempotent residuated lattice $\mathbf{A}$ is a unital band and the relation on A defined by $a \sqsubseteq b \iff a \cdot b = a$ is a preorder that we call the *monoidal preorder* of $\mathbf{A}$; if the product of $\mathbf{A}$ is also commutative, then $\langle A, \sqsubseteq, e\rangle$ is a unital meet-semilattice with order $\sqsubseteq$ and greatest element e. An idempotent residuated lattice is *conservative* if its monoidal preorder is total, that is, for all a, b in A, $a \cdot b \in \{a, b\}$. For instance, every idempotent residuated chain is conservative.

In the present work, we complete the study of the class of conservative commutative residuated lattices initiated in [1] — in which we gave an account of its finite members only —, and present a general structure theory for all the members of this class. We show that the lattice of every conservative commutative residuated lattice can be described as a tree in which, moreover, all its leaves are also linearly ordered.

More in detail, a *conservative tree* is a first-order structure $\mathbf{M} = (M, \sqcap, \trianglelefteq, e)$, where

(1) $(M, \sqcap)$ is a meet-semilattice that is also a tree (we will denote its order by $\sqsubseteq$ and the set of its maximal elements by M^+);
(2) every element of M is below a $\sqsubseteq$-maximal element;
(3) $(M^+, \trianglelefteq)$ is a chain with bottom element e;
(4) for every $m \in M$, the set $\nabla m := \{p \in M^+ : m \sqsubseteq p\}$ is a closed interval of $(M^+, \trianglelefteq)$.

We prove that every conservative commutative residuated lattice $\mathbf{A}$ gives rise to a conservative tree $\mathbf{M_A}$; and that from every conservative tree $\mathbf{M}$ we can construct a conservative commutative residuated lattice $\mathbf{A_M}$. Moreover, these correspondences are inverse to each other.

We use this representation to settle various open problems. In particular, we show that local finiteness fails for the class of conservative commutative residuated lattices, as it contains a 1-generated infinite member. We prove also that the class of conservative commutative residuated lattices has the strong amalgamation property constructing a strong amalgam for every V-formation.

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Constructing conical and perfect residuated lattices

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Residuated structures play an important role in the field of algebraic logic since they constitute the equivalent algebraic semantics, in the sense of Blok and Pigozzi, of substructural logics (see [2, 3]). These encompass many of the interesting nonclassical logics: intuitionistic logic, fuzzy logics, relevance logics, linear logics and also classical logic as a limit case. Thus, the algebraic investigation of residuated lattices is a powerful tool in the systematic and comparative study of such logics. While many deep results have been obtained in the last decades, the multitude of different kinds of residuated lattices and their rich theory makes their study fairly complicated, and at the present moment large classes of residuated lattices lack a structural description. Because of this, the development of constructions that allow one to obtain new structures from known ones is of utter importance for the understanding of both residuated lattices and substructural logics as a whole.

In this contribution we introduce two constructions that, starting from two algebras, obtain respectively a conical commutative residuated lattice and a perfect (bounded) commutative residuated lattice; the underlying idea is on one side to generalize the constructions in [4] (used by the authors to provide a description of conical idempotent residuated lattices), and on the other side to generalize the generalized disconnected rotation construction (see e.g. [1]).

To be more clear let us first recall some important definitions. A *commutative residuated lattice* is an algebra $\mathbf{A} = (A, \cdot, \to, \wedge, \vee, 1)$ of type $(2, 2, 2, 2, 0)$ where: $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a commutative monoid, and the residuation law holds, i.e. $x \cdot y \leq z$ if and only if $y \leq x \to z$ for any $x, y, z \in A$. A residuated lattice $\mathbf{A}$ is: *integral* if it has a maximum element which coincides with the monoidal unit 1; *conical* if the unit 1 is a conical element, i.e. for each $a \in A$, either $a \leq 1$ or $1 \leq a$. Moreover, we call *bounded* an integral commutative residuated lattice with an extra constant 0 that is the smallest element; we call a bounded commutative residuated lattice *perfect* if it can be seen as the disjoint union of a congruence filter F and the set $\{a \in F : x \to 0 \in F\}$.

Let us now discuss the wanted constructions. We start from a commutative integral residuated lattice $\mathbf{A}$ and an algebra $\mathbf{B}$ with some properties. To be more precise, let us endow both $\mathbf{A}$ and $\mathbf{B}$ with a closure operator γ such that $\gamma(x)$ is a conical and idempotent element for any x. Let then $\gamma[\mathbf{A}]$ and $\gamma[\mathbf{B}]$ be the set of its fix points; one can observe that the set of elements having the same γ-image can be seen as a bubble with the γ-fixed point as top (see the image below for a pictorial intuition). Hence let one start form a commutative integral residuated lattice $\mathbf{A}$ which is the ordinal sum of its bubbles and from an algebra $\mathbf{B}$ such that any $x \in B$ is above 1 and $\mathbf{B} \cup \{\bot, \top\}$ is a residuated lattice where the product between elements of different bubbles is the join. Let then $\gamma_{\mathbf{A}} : \mathbf{A} \to \mathbf{A}$ and $\gamma_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}$ be such maps.

Observe that $\gamma_{\mathbf{A}}[\mathbf{A}]$ and $\gamma_{\mathbf{B}}[\mathbf{B}]$ are isomorphic; let us denote with $'$ such isomorphism and its inverse.

We will now define new operations on the domain $\mathbf{A} \cup \mathbf{B}$; in order to do so, we extend the map $'$ to be a map from $\mathbf{A} \cup \mathbf{B}$ to $\mathbf{A} \cup \mathbf{B}$ is a way such that: for any $x \in A$, $x' = (\gamma_{\mathbf{A}}(x))'$ and for any $x \in B$, $x' = (\gamma_{\mathbf{B}}(x))'$. We call such a map $'$ a *complementation* of $\mathbf{A}$ and $\mathbf{B}$.

Now, for the first construction the idea is to copy $\mathbf{B}$ above $\mathbf{A}$, and consider a new product that extends the one of $\mathbf{A}$ and $\mathbf{B}$ in the following way: if $x \in A$ and $y \in B$ $x \cdot y = x$ if $x \leq y'$, and $x \cdot y = y$ otherwise. We show that such product is residuated and yields a conical residuated lattice, $\mathbf{C}_{\mathbf{A},\mathbf{B}}$. For the second construction, we instead copy $\mathbf{B}$ below $\mathbf{A}$; the product remains the old one on $\mathbf{A}$, $x \cdot y = x$ if $x \in B, y \in A$ and $x' > y$, and it is 0 in all other cases. We demonstrate that such product is residuated, and obtain the corresponding perfect residuated lattice, $\mathbf{P}_{\mathbf{A},\mathbf{B}}$. The following figure sketches the two constructions.

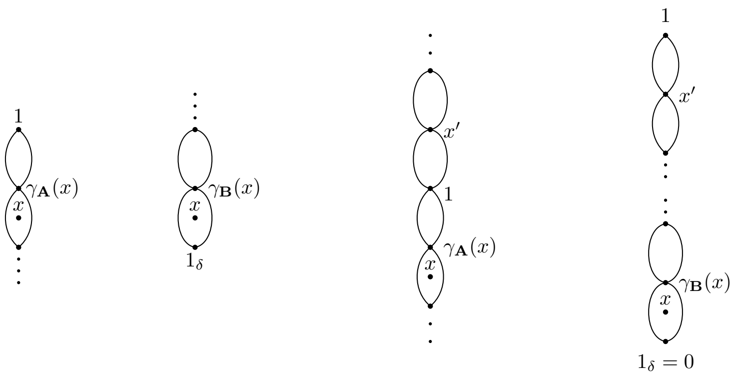

Figure 1: From the left: $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}_{\mathbf{A},\mathbf{B}}$, $\mathbf{P}_{\mathbf{A},\mathbf{B}}$.

Starting from these constructions, we develop and study a connection between subclasses of conical and perfect commutative residuated lattices.

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Topological Duality for Distributive Lattices: Theory and Applications

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This is a talk about the book [1] on topological duality theory for bounded distributive lattices recently published by Cambridge University Press, and it will be presented jointly by the authors. The purpose of the talk will be to give an overview of the content and potential uses of the book in teaching and research, and to sound out the audience on potentially useful additional resources we could put on the web. In the rest of this abstract, we draw from the book's preface to give a quick overview of its contents as we plan to present them in the conference presentation.

The book is a course on Stone-Priestley duality theory, with applications to logic and the foundations of computer science. Our target audience includes both graduate students and researchers in mathematics and computer science. The main aim of the book is to equip the reader with the theoretical background necessary for reading and understanding current research in duality and its applications. We have aimed to be didactic rather than exhaustive, while we did give technical details whenever they are necessary for understanding what the field is about.

A unique feature of the book is that, in addition to developing general duality theory for distributive lattices, we also show how it applies in a number of areas within the foundations of computer science, namely, modal and intuitionistic logics, domain theory and automata theory. The use of duality theory in these areas brings to the forefront how much their underlying mathematical theories have in common. It also prompts us to upgrade our treatment of duality theory with various enhancements that are now commonly used in state-of-the-art research in the field. Most of these enhancements make use of *operators* on a distributive lattice: maps between lattices that only preserve part of the lattice structure. We give a textbook exposition of the theory of lattices with operators, and dualities for them, as it was developed in the second half of the 20th century. Our exposition of the theory also treats several of its by now classical applications, such as those to free distributive lattices, quotients and subspaces, implication-type operators, Heyting algebras and Boolean envelopes.

In the first chapters of the book, we keep the use of category theory to a minimum. We then set the results in the more abstract and general framework of category theory. This development also allows us to show how Priestley's duality fits well in a more general framework for the interaction of topology and order, which had been developed by Nachbin shortly before. We show how the various classes of topological spaces with and without order, introduced by Stone, Priestley and others, all relate to each other, and how they are in duality with distributive lattices and their infinitary variant, frames.

The book ends with an extended exposition of two more modern applications of duality theory to theoretical computer science, namely to domain theory and to automata theory. The domain theory that we develop is organized around three separate results: Hoffmann-Lawson duality; the characterization of those dcpos and domains, respectively, that fall under Stone duality; and Abramsky's celebrated 1991 Domain Theory in Logical Form paper. The duality-theoretic approach to automata theory that we develop in the book originates in work due to the first author with Grigorieff and Pin. It is organized around a number of related results, namely:

finite syntactic monoids can be seen as dual spaces, and the ensuing effectivity of this powerful invariant for regular languages; the free profinite monoid is the dual of the Boolean algebra of regular languages expanded with residuation operations and, more generally, topological algebras on Boolean spaces are duals of certain Boolean algebras extended by residual operations. As an extended application example, we use duality to give a profinite equational characterization for the class of piecewise testable languages; and we end by discussing a characterization of those profinite monoids for which the multiplication is open.

These two applications, and in particular the fact that we treat them in one place, as applications of a common theory, are perhaps the most innovative and special aspects of this book. Domain theory is the most celebrated application of duality in theoretical computer science and our treatment is entirely new. Automata theory is a relatively new application area for duality theory and has never been presented in textbook format before. More importantly, both topics are currently at the forefront of active research seeking to unify semantic methods with more algorithmic topics in finite model theory. While previous treatments remained focused on the point of view of domains/profinite algebra, with duality theory staying peripheral, a shared innovative aspect of the presentations of these topics in this book is that both are presented squarely as applications of duality.

Finally, a completely original contribution of this book, which emerged during its writing, precisely thanks to our treatment of the two topics as an application of a common theory, is the fact that a notion we call "*preserving joins at primes*" turns out to be central in both the chapter on domain theory and in that on automata theory. This notion was introduced in the context of automata theory and topological algebra by the first author in 2016; its application to domain theory is new to this book and reflects a key insight of Abramsky's Domain Theory in Logical Form. We believe this point to be an exciting new direction for future research in the field that we hope some readers of the book will be inspired to take up.

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`MUltseq` 2.0
A general purpose finite-valued prover

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1 Introduction

The system `MUltlog` was developed and implemented in Prolog in Vienna and its theoretical foundations can be found in the thesis dissertation of R. Zach ([10] and also [3, 2]). The main motivation behind `MUltlog` is that from any finite-valued logic first order logic it is always possible to generate a set of rules for m-sequents (similar to Rousseau's approach, [9]) that is sound and complete for this logic and enjoys the cut elimination property.

These systems, restricted to the propositional case, were studied from the algebraic logic point of view in Barcelona (see [6, 8]) and a new set of algorithms called `MUltseq` was produced and implemented, in such a way that the rules produced by `MUltlog` were used to generate proofs in the different logical or algebraic systems naturally associated with the original finite-valued logic (see [5, 7]). Later on, due to the high compatibility of the systems, `MUltlog` and `MUltseq` were officially "married" ([1]).

In its current state `MUltseq` is able to produce proofs of formulas (or give counterexamples) and to determine if a consequence relation is valid in an arbitrary finite-valued logic, and also to check if an equation or quasi-equation is valid in an arbitrary finite algebra, but these formulas or equations must be previously introduced to the system by the user.

2 `MUltseq` 2.0

Recent changes made to `MUltlog`, now available on GitHub [11], suggested some new and major improvements to `MUltseq`. We are happy to announce `MUltseq` 2.0, a new Prolog system that expands the capability of the previous one. We will show how to automatically produce a scientific paper with a comprehensive study of the properties of a given finite-valued logic. The logic is understood as a finite algebra with a set of propositional connectives and a set of designated truth values (and a set of anti-designated values), plus the rules obtained by `MUltlog`. More precisely, the "paper" will contain:

1. the description of the logic and its connectives, as well as the rules produced by `MUltlog`;
2. for each connective or subset of connectives, a checklist of the usual properties they may have (commutativity, associativity, idempotency, ...);
3. a list of valid formulas (tautologies) in the logic;
4. a list of equations (quasi-equations) valid in the algebra and in the variety (quasi-variety) it generates;

5. a checklist of valid entailments in various consequence relations.

The consequence relations that can be considered are preservation of designated truth values and, if the set of truth values is ordered, preservation of degrees of truth. If anti-designated truth values are provided, it can also analyse the corresponding strict/tolerant and tolerant/strict relations [4]. Examples of entailments to check are Modus Ponens and De Morgan laws, and other properties relevant for the algebraic study of the logic.

In the conference we will present the main results that make `MUltseq` work and a live tool demo will be organised to generate papers for different logics.

It goes without saying that in every case there is a limitation on the length/depth of the formulas and on the number of premises that can be easily adapted. Users will still be able to choose special objects (formulas, inferences, equations ...) that will appear as designated in the paper, possibly with the corresponding proofs or counterexamples like in the previous version of `MUltseq`. Finally, the results obtained may be stored in a Prolog database for further investigations and comparison of different logics.

We hope this system will help to simplify calculations and serve as a kind of useful general purpose calculator for finite-valued logics in the propositional case. (See [12] for a recent example of such an application.)

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On abstract model theory and logical topologies*

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The intersection of mathematical logic and topology has been a vibrant area of research for several decades. McKinsey and Tarski's seminal work on the topological interpretation of modal logic [11, 12], played a crucial role in establishing the relationship between logic and topology. Additionally, McKinsey and Tarski developed an algebraic and topological framework for Intuitionistic Logic (IL) and Modal Logics (ML) [12, 13], demonstrating that topological spaces can serve as interpretation models for IL and ML. Tarski further proved that S4 is complete with respect to topological spaces [16], while McKinsey and Tarski showed that S4 is the modal logic of real numbers in 1944 [12]. Various articles offer additional insights into these topics [3, 2, 14], and alternative approaches can be found in the works of Lawvere [8] and Goldblatt [5]. In Universal Logic, Lewitzka presents a different approach to logical systems [9, 10], constructing a theory of logical representations (a logic map) to leverage the fact that every logical system can define a topology within its theory set. In this abstract we propose the study of logic and topology independently from the underlying logic.

Our approach is based on the theory of institutions. Institutions constitute the main branch of the categorical abstract model theory, which formalizes the notion of a logical system, including syntax, semantics and the satisfaction relation between them [1]. An *institution* $\mathcal{I} = \left(\mathbb{S}ig^{\mathcal{I}}, \boldsymbol{Sen}^{\mathcal{I}}, \boldsymbol{Mod}^{\mathcal{I}}, \models^{\mathcal{I}}\right)$ consists of (a) a category $\mathbb{S}ig^{\mathcal{I}}$, the objects of which are called *signatures*; (b) a functor $\boldsymbol{Sen}^{\mathcal{I}} : \mathbb{S}ig^{\mathcal{I}} \to \mathbb{S}et$ such that it assigns a set the elements of which are called *sentences* over each signature; (c) a functor $\mathbf{Mod}^{\mathcal{I}} : \left(\mathbb{S}ig^{\mathcal{I}}\right)^{op} \to \mathbb{CAT}$ giving a category the objects of which are called Σ-*models* and the arrows of which are called Σ-*morphisms* for each signature Σ, and (d) a relation $\models^{\mathcal{I}}_{\Sigma} \subseteq \left|Mod^{\mathcal{I}}(\Sigma)\right| \times \boldsymbol{Sen}^{\mathcal{I}}(\Sigma)$ for each $\Sigma \in \left|\mathbb{S}ig^{\mathcal{I}}\right|$, called Σ-*satisfaction* such that for each morphism $\phi : \Sigma \to \Sigma'$ in $\mathbb{S}ig^{\mathcal{I}}$, the *satisfaction condition*.

For every signature Σ we define a class of topologies over the category of models $\mathbf{Mod}(\Sigma)$ based on the class of subsets of $\boldsymbol{Sen}(\Sigma)$, this class being closed under union. We consider the morphisms of model categories $\mathbf{Mod}(\phi) : \mathbf{Mod}(\Sigma') \to \mathbf{Mod}(\Sigma)$ induced by signature morphisms $\phi : \Sigma' \to \Sigma$ standing for change of notation. We investigate the broader possible class of topologies in $\mathbf{Mod}(\Sigma)$ whose members are mapped to topologies in $\mathbf{Mod}(\Sigma')$ via the arrow $\mathbf{Mod}^{-1}(\phi)$, as well as the broader possible class of topologies in $\mathbf{Mod}(\Sigma')$ whose members are mapped to topologies in $\mathbf{Mod}(\Sigma)$ via $\mathbf{Mod}(\phi)$ [7]. Furthermore, we investigate under which circumstances (ie additional properties of such morphisms) an additional structure of topologies is preserved. The questions that arise from this inquiry are on the model theoretic properties of these topologies. We prove several theorems in this direction, such as that the class of topologies includes topologies defined over categories of elementary equivalent models (ie. models that satisfy the same sentences), and the essential link between elementary equivalent models and the intersection preserving properties. Finally, we attempt to generalize our inquiry to Grothendieck topologies, given institutions with the appropriate categorical properties.

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These findings contribute and generalize the results from [6, 15] and new results from [4]. In [6], the author has introduced the notion of topological semantics in the framework of abstract model theory through institution-independent theory. Within this framework, semantic completeness can be explored through topological concepts.

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Unification for temporal logic via duality and automata

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The *unifiability problem* for an equational theory $\mathcal{E}$ is the decision problem that asks, given as input two $\mathcal{E}$-terms

$$A(x_1, \ldots, x_n) \quad \text{and} \quad B(x_1, \ldots, x_n) \ , \tag{1}$$

to decide whether or not there exists a syntactic substitution of the variables by terms, $x_i \mapsto C_i$, such that the resulting terms $A(C_1, \ldots, C_n)$ and $B(C_1, \ldots, C_n)$ are equal modulo the theory $\mathcal{E}$.

We establish decidability of the unifiability problem for a logic we call $\mathbf{X}$, the next-fragment of linear temporal logic, enriched with an arbitrary fixed number of propositional constants $\mathbf{c}_1, \ldots, \mathbf{c}_n$. In algebraic terms, the equational theory $\mathcal{E}_\mathbf{X}$ associated with the logic $\mathbf{X}$ is the theory of Boolean algebras with an arbitrary fixed number of nullary function symbols, $\mathbf{c}_1, \ldots, \mathbf{c}_n$, and a unary function symbol, X, which denotes a Boolean endomorphism. Our main result is that the equational theory of the free algebras of this variety is decidable. In the remainder of this abstract, we will give an overview of our approach for proving this result.

Let Σ be a finite alphabet. The *de Bruijn graph* $B_d = (\Sigma^d, S_d)$ of dimension $d \geq 1$ is the graph with set of vertices Σ^d and Σ-colored edge relation defined as

$$S_d \coloneqq \{(bv, a, va) \ : \ a, b \in \Sigma, v \in \Sigma^{d-1}\} \subseteq \Sigma^d \times \Sigma \times \Sigma^d \ .$$

One may think of the de Bruijn graph as a deterministic automaton that 'remembers' the d letters that were most recently read. We define the *de Bruijn graph mapping problem* to be the decision problem that asks, for an input graph with Σ-coloring on the edges, $G = (V_G, E_G)$, whether or not there exist $d \geq 1$ and a homomorphism (i.e., colored-edge-preserving function) from B_d to G.

Theorem 1. *The unifiability problem for $\mathcal{E}_\mathbf{X}$ is computationally equivalent to the de Bruijn graph mapping problem.*

The proof of Theorem 1 uses Stone duality and a step-wise construction for the free $\mathcal{E}_\mathbf{X}$-algebra, as we will explain further below. Given this result, our new goal is to show that the de Bruijn graph mapping problem is decidable. For this, we introduce two notions on graphs, that we call *cycle-connected* and *power-connected*, and we prove:

Theorem 2. *A graph G has a cycle- and power-connected subgraph if, and only if, there exist $d \geq 1$ and a homomorphism $B_d \to G$.*

Since we also show that it can be checked (in exponential time) whether or not a graph has a cycle- and power-connected subgraph, Theorem 2 in particular implies that the de Bruijn graph mapping problem is decidable, from which the decidability of unifiability in $\mathcal{E}_\mathbf{X}$ then follows. Since the reduction from an instance of unifiability to an instance of de Bruijn graph mapping in Theorem 1 takes at most exponential time, our algorithm as a whole gives a 2-EXPTIME upper bound.

*This abstract draws from recently submitted and ongoing joint work with Johannes Marti (University of Zürich) and Michelle Sweering (CWI, University of Amsterdam). We thank Stéphane Desarzens, George Metcalfe, Antoine Mottet, and Leif Sabellek for many inspiring discussions.

To explain a bit more about the proof of Theorem 1, let us first stratify the set of $\mathcal{E}_{\mathbf{X}}$-terms by *depth*, i.e., the maximum nesting of function symbols occurring in a term. For any $d \geq 0$, there are, up to equivalence, only finitely many candidate unifiers *of depth* $\leq d$. Writing $\mathbb{A}_d$ for the set of equivalence classes of ground $\mathbf{X}$-formulas of depth $\leq d$, we thus obtain a chain of inclusions of finite sets

$$\mathbb{A}_0 \hookrightarrow \mathbb{A}_1 \hookrightarrow \mathbb{A}_2 \hookrightarrow \cdots \ . \tag{2}$$

Each of the finite sets $\mathbb{A}_d$ carries the syntactic structure of a Boolean algebra, and, for each $d \geq 0$, a Boolean algebra homomorphism,

$$\mathrm{X}_d^{\mathbb{A}} : \mathbb{A}_d \to \mathbb{A}_{d+1} \ ,$$

which sends the equivalence class of a formula ϕ of depth $\leq d$ to the equivalence class of $\mathrm{X}\phi$. The chain (2) decomposes the initial algebra for $\mathcal{E}_{\mathbf{X}}$ as a colimit of a chain of finite algebras. Now, following a methodology pioneered in [3], we apply finite Stone duality to the diagram (2), in order to obtain a dual diagram in the category of finite sets, namely, an inverse chain

$$V_0 \twoheadleftarrow V_1 \twoheadleftarrow V_2 \twoheadleftarrow V_3 \twoheadleftarrow \cdots \tag{3}$$

where $V_d \coloneqq \mathrm{At}\,\mathbb{A}_d$, the set of *atoms* of the Boolean algebra $\mathbb{A}_d$. Using duality, we show that the operation X also gives rise to a graph structure on the set V_d, which makes it isomorphic to B_d, the de Bruijn graph of dimension d. A further application of Stone duality to the possible solutions of a unification problem then shows that a unifying substitution one-to-one corresponds to a graph homomorphism from one of the graphs V_d to a graph G, which can be computed within exponential time from the formulas to be unified.

In order to prove Theorem 2, we significantly extend a number of existing results from the literature [2, 1]. There, the restriction of the de Bruijn graph mapping problem to *deterministic* target graphs was shown decidable, by characterizing the deterministic homomorphic images of de Bruijn graphs as precisely those graphs which are strongly connected and *d-synchronizing*. The latter condition says that, for every $w \in \Sigma^d$, there is a node y_w such that for every $x \in V_G$, there exists a path $x \xrightarrow{w} y_w$ in G.

However, the homomorphic image of a deterministic graph, such as B_d, may fail to be deterministic, and Theorem 1 implies that essentially any graph, not necessarily deterministic, can occur as the graph associated with a unification problem for the logic $\mathbf{X}$. Our definitions of 'power-connected' and 'cycle-connected' capture properties that generalize the d-synchronizing condition to the non-deterministic setting, in two distinct directions. We designed the conditions in such a way that any graph admitting a homomorphism from a de Bruijn graph must satisfy both conditions. The most difficult combinatorial part of our work lies in the converse direction of Theorem 2. A crucial idea there is that of *minimizers*, which originates in the literature on string compression algorithms [6, 5]. This notion allows one, in any non-highly-periodic word $w \in \Sigma^d$ for large d, to single out a particular position in the word w which remains stable when walking from w in any direction in the de Bruijn graph B_d during at most r steps.

The work we describe here instantiates a general (co-)algebraic approach towards unification, which we plan to develop in further work. We hope that this will allow us to delineate the precise scope of the method that we followed here, addressing in particular the question of whether or not it can be helpful for the open problem of decidability of unifiability in basic modal logic $\mathbf{K}$. The analogous result to Theorem 1 for $\mathbf{K}$ was stated in [4], but the corresponding combinatorial problem on hypergraphs is currently out of our reach. Further questions for future work include whether the 2-EXPTIME upper bound on unifiability in $\mathbf{X}$ is tight, and how difficult it is to actually compute unifiers, if they exist: Our current method only gives a quadruple-exponential bound, but we expect that a more syntactic analysis of the problem could improve on this.

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Generalised Unions of Conjunctive Queries in the Algebraic Data Model

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Recent work in categorical database theory [Spi12, SW15, SSVW17, SW17] develops a new data model which we refer to as the *algebraic data model* because it is based on algebraic (i.e. Lawvere) theories, in contrast to both the traditional relational data model [AHV95] and to other approaches to categorical database theory, such as finite limit sketches [RW91] [JR02] or attributed C-sets [PLF22]. The algebraic data model builds on the insight that categories can be interpreted as database schemas, by using objects to represent tables and outgoing arrows to represent columns which refer to other tables (i.e. foreign keys). That is, a database instance on a schema $\mathcal{C}$ is defined as a copresheaf $\mathcal{C} \to \mathbf{Set}$, which populates each table with a set of rows, and given any other schema $\mathcal{D}$, functors $\mathcal{C} \to \mathcal{D}$ serve as the basis for defining data migration operations.

The approach just sketched is enticing, because "every theorem about small categories becomes a theorem about databases" [Spi12]. However it turns out to be too naive to formalize uses of databases beyond simplistic querying. In particular, in the context of data integration some values may be treated up to isomorphism, such as the labelled nulls generated during integration to fill in missing values, while values that appear in the input data, such as the names of people and their salaries, must be treated up to equality. In this sense, categories and copresheaves have too many automorphisms to be used in data integration. This *attribute problem* is solved by defining values to be expressions in an algebraic theory [SSVW17, SW17]. To this end, schemas are defined by equipping categories with an algebraic profunctor into a fixed algebraic theory, called the *typeside*, whose objects represent data types (e.g. Nat, String) and whose morphisms represent data operations (e.g. addition, append). The typeside adds the necessary rigidity for data integration by requiring that constants (such as "Alice") but not variables (such as Alice's unknown salary n) be preserved by morphisms of instances. Algorithms follow by specialising definitions to finite presentations of schemas, instances, etc. and are implemented in the open-source CQL tool available at categoricaldata.net.

Current research about the algebraic data model has progressed from studying the data transformations associated with *functors* to studying the more structured data transformations associated with *profunctors*, which we call *proqueries*. Proqueries, which subsume traditional conjunctive queries, have been referred to as *bimodules* in [SSVW17], where the theory is developed in the form of a proarrow equipment. Proquery presentations, in turn, were developed in [SW17] under the name of *uberflowers*. Evaluating a proquery on a database instance according to its conjunctive query semantics produces another instance on a different schema, and this process is sufficient for querying. However, it is also possible to *coevaluate* a query, which constitutes the left adjoint to the evaluation functor just described. In fact, the evaluation-coevaluation adjunction is a particular case of the general geometric realisation/nerve adjunction, see [SSVW17, Remark 8.8]. Unlike evaluation, which cannot create labelled nulls in its

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output, coevaluation can, and thus allows us to perform data integration. We are currently studying the exact expressive power of query coevaluation compared to existing data integration techniques. Finally, the connection between proqueries and relational queries gives us a way to implement *query migration*: given an $\mathcal{A}$-shaped proquery $Q : \mathcal{A} \nrightarrow \mathcal{B}$ on schema $\mathcal{B}$, we can migrate Q along a proquery $F : \mathcal{B} \nrightarrow \mathcal{C}$ by composition, i.e. $F \circ Q$ is the desired migrated query. At the level of proquery presentations, composition is implemented by an operation which is called *view unfolding* in the relational literature [KP18]. Kan lifts and extensions of proqueries also have uses in data exchange operations and are under current research.

In this talk we will explain the algebraic data model and present our work on a new generalisation of proqueries which is analogous to *unions* of conjunctive queries. We call this generalisation *praqueries* since it is known that, when the typeside is trivial, they are equivalent to parametric right adjoint functors, also known as prafunctors. We give a composition law for praqueries, hence generalising the proarrow equipment structure in [SSVW17]. Importantly, we also develop the theory of praquery presentations, including an effective algorithm for composition of finite praquery presentations which generalises the view unfolding algorithm. We are able to obtain a correctness proof for this algorithm, which constitutes our main result and the capstone of our talk.

Along the way, we correct some technical errors in the literature, and we provide a proof of correctness for composition of proquery presentations, which to the best of our knowledge was not present in the literature. We expect to settle some further conjectures about praqueries, namely that praqueries are equivalently prafunctors between categories of instances which preserve type-algebras, and the issue of existence of right Kan lifts in the bicategory of praqueries.

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On Free Generalized 3-valued Post algebras

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Abstract

We develop the theory of generalized 3-valued Post algebras (P_3^ω-algebras), which are obtained from Komori type S_2^ω-algebras by enriching its signature with the constant $\frac{1}{2}$. Finitely generated free and projective algebras are described in the variety $\mathbf{P}_3^\omega$ generated by P_3^ω-algebras. The variety $\mathbf{P}_3^\omega$ contains only one proper subvariety – the subvariety of 3-valued Post algebras [5].

1 P_3^ω-algebras

We introduce new class $\mathbf{P}_3^\omega$ of generalized 3-valued Post algebras that form a variety. P_3^ω-algebra is a system $(A, \vee, \wedge, \oplus, \otimes, \neg, 0, \frac{1}{2}, 1)$, where A is a nonempty set of elements, 0, $\frac{1}{2}$, and 1 are distinct constant elements of A, and $\vee, \wedge, \oplus, \otimes$ are binary operations on elements of A, and $\neg$ is a unary operation on elements of A, obeying a finite set of axioms (identities).

The algebra $(A, \vee, \wedge, \oplus, \otimes, \neg, 0, \frac{1}{2}, 1)$ is P_3^ω-algebra if $(A, \oplus, \otimes, \neg, 0, 1)$ is an S_2^ω-algebra (defined by Komori in [3]), i. e. MV-algebra satisfying the identity $(3(x^2))^2 = 2(x^3)$, and $(A, \vee, \wedge, 0, 1)$ is a distributive bounded lattice satisfying the following identities: $\frac{1}{2} \oplus \frac{1}{2} = 1$, $\frac{1}{2} \otimes \frac{1}{2} = 0$, $\frac{1}{2} \otimes (x \wedge \neg x) = 0$, $\frac{1}{2} \oplus (x \vee \neg x) = 1$, $\neg \frac{1}{2} = \frac{1}{2}$. The algebra $(\{0, \frac{1}{2}, 1\}, \vee, \wedge, \oplus, \otimes, \neg, 0, \frac{1}{2}, 1)$ with the following operations: $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$, $x \oplus y = \min(1, x + y)$, $x \otimes y = \max(0, x + y - 1)$, $\neg x = 1 - x$, is an example of 3-valued Post algebra. Notice, that this algebra is obtained by enriching the signature of an MV_3-algebra S_2 [2] with the constant $\frac{1}{2}$. Moreover, the algebra $(\{0, \frac{1}{2}, 1\}, \vee, \wedge, \oplus, \otimes, \neg, 0, \frac{1}{2}, 1)$ is functionally equivalent to the 3-element Post algebra P_3. Indeed, it is enough to express the cyclic negation $\sim x = (\frac{1}{2} \otimes x) \vee (\neg x \otimes \neg x)$.

MV-algebras are the algebraic counterpart of the infinite valued Lukasiewicz sentential calculus, as Boolean algebras are concerning the classical propositional logic. In contrast with what happens for Boolean algebras, some MV-algebras are not semi-simple, i.e. the intersection of their maximal ideals (the radical of A) is different from $\{0\}$. The simple example of non semi-simple MV-algebra is given by C. Chang in [1] (the algebra C). The MV-algebras generated by their radical are called perfect.

Mundici [4] defined correspondence functor Γ between MV-algebras and lattice-ordered abelian groups (abelian l-groups) with strong unit, and proved that Γ is a categorical equivalence. We define analogical functor Γ_c of P_3^ω- algebras and l-groups with strong unit u. More precisely, for every abelian l-group G, the functor Γ_c equips the unit interval $[0, 2u]$ with the operations: $x \vee y = max(x, y), x \wedge y = min(x, y), x \oplus y = 2u \wedge (x+y), x \otimes y = 0 \vee (x+y-2u), \neg x = 2u - x, 1 = 2u$.

Notations.
(i) $D_0 = \Gamma(Z, 2) \cong P_3$, with 1 as a strong unit.
(ii) $D_1 = D = \Gamma_c(Z \times_{lex} Z, (2, 0))$ with the strong unit $(1, 0)$, the generator $d_1 (= (0, 1))$, and $\times_{lex}$ is the lexicographic product.

(iii) $D_m = \Gamma_c(Z \times_{lex} ... \times_{lex} Z, (2,0,...,0))$ with the strong unit $(1,0,...,0)$, the generators $d_1(= (0,0,...,1)), ..., d_m(= (0,1,...,0))$, where the number of factors of Z is equal to $m+1$.

(iv) Let D^*_m be the subalgebra of D_m generated by the radical (intersection of all maximal ideals) of D_m, where $m \in Z^+$.

Proposition: Let G be an abelian l-group with the strong unit u. Then $\Gamma_c(G, 2u)$ is a generalized P^ω_3-algebra $([0, 2u], \vee, \wedge, \oplus, \otimes, \neg, 0, u, 2u)$.

A subset F of a P^ω_3-algebra A is said to be an *ideal* if 1) $0 \in I$, 2) if $x, y \in I$, then $x \oplus y \in I$, and 3) if $x \in I$ and $y \leq x$, then $y \in I$.

Theorem:

1) D generates the variety $\mathbf{P}^\omega_3$.

2) There exists lattice isomorphism between the lattice of ideals of a P^ω_3-algebra A and the lattice of congruences of a P^ω_3-algebra A.

3) m-generated free P^ω_3-algebra is isomorphic to ${D^*_m}^{3^m}$.

5) The P^ω_3-algebras P_3 and D^m are projective for every $m \in Z^+$.

6) The variety $\mathbf{P}_3$ of 3-valued Post algebras is the only proper subvariety of the variety $\mathbf{P}^\omega_3$.

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Semantics for Non-symbolic Computation: Neural Networks and Other Analog Computers

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Introduction Despite the great technological progress, we are lacking a foundational theory of modern artificial intelligence (AI). Specifically, we want to interpret, explain, and verify the 'sub-symbolic' computation performed by neural networks that drive this success. For classical 'symbolic' computation, this problem was solved by *semantics*: the mathematical description of the meaning of program code. In this talk, we develop one approach to an analogous semantics for non-symbolic computation performed by neural networks and other analog computers. To do so, we first summarize the three semantics for symbolic computation, and then we describe our analogous components—systems, domains, and logic—for non-symbolic computation, visualized in figure 1. The key idea is to represent the dynamics of the non-symbolic computation as a limit of symbolic approximations, which are given by observations.

Semantics for symbolic computation Symbolic computation is specified by some *program code* P written in some programming language, and semantics should assign meaning to P. There are three approaches. First, 'systems': *Operational semantics* describes P by the steps a machine would take to implement this program, so the meaning of our program is given by a transition system. Second, 'domains': *Denotational semantics* describes P by the function (or denotation) $[\![P]\!]$ that it computes and the finite approximations to this function. The set of all denotations and approximations of programs of a given type σ forms a so-called domain D_σ. Third, 'logic': *Logical semantics* describes P by the properties it has: e.g., if the input is 1, then executing P yields an even output, which is written as the Hoare triple $\{\mathsf{is}\ 1\}P\{\mathsf{is\ even}\}$.

Ideally, these three semantics are in harmony: *Partial correctness* requires that if a Hoare triple $\{\varphi\}P\{\psi\}$ is provable, then running program P in a state satisfying φ results in a state satisfying ψ (if P terminates). *Full abstraction* requires that two programs have the same denotation iff the machines running the two programs show the same behavior. *Stone duality* requires that the properties of P jointly determine the denotation $[\![P]\!]$, and vice versa [1].

For an analogous semantics for non-symbolic computation, we now explicate the italic terms.

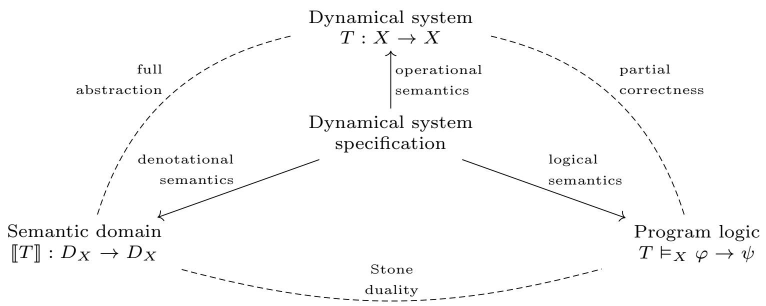

Figure 1: The threefold semantics for non-symbolic computation.

Systems If symbolic computation is specified by program code, how is non-symbolic computation specified? The answer is: by dynamical systems [2]. Let's consider neural networks as an example. (Other famous examples are cellular automata or differential analyzers.) Understanding their training dynamics is essential for a theory of deep learning. Neural networks are trained by backpropagation: given a batch of training data, it updates the current weights w of the network to weights w' with smaller training loss. Hence backpropagation specifies a dynamical system $T : W \times D^\omega \to W \times D^\omega$, where W is the weight space and D is the set of batches of data, and T maps a pair (w, d) of a weight w and a sequence of batches d to the pair $(w', \sigma(d))$, where w' is the result of updating w with the batch d_0 and $\sigma(d) = d_1 d_2 \ldots$ (i.e., σ is the shift operator on sequences). Thus, the analogue of a program code is a dynamical system specification like backpropagation (or the rule of a cellular automaton or the differential equation specifying the differential analyzer, etc.). The analogue of a transition system is a dynamical system $T : X \to X$. Formally, we take X to be a zero-dimensional compact Polish space and T a continuous function (as studied in the field of topological dynamics in dimension zero [3]; though in [4] we also cover probabilistic systems).

Domains As in symbolic semantics, we obtain the 'meaning' of the dynamical system in the limit of finite, 'interpretable' approximations to the system. These approximations are given via observations about the system: e.g., that with the current set of weights w the neural network classifies this given image correctly. As will be explained in the talk, we package these observations as finite domains and, by refining the observations, we obtain in the limit (in the category-theoretic sense) a domain D_X with a Scott-continuous function $[\![T]\!] : D_X \to D_X$, which we call the dynamical domain. Formally, we develop this idea as a functor from the category of dynamical systems to the category of dynamical domains. This functor has a left adjoint, which naturally restricts to an equivalence—this can be regarded as a form of full abstraction [4].

Logic Finally, the finite observations of the system can be identified with clopen subsets of the state space X. The Hoare triple $\{\varphi\}T\{\psi\}$ then says: whenever we observe the system having property φ now, we observe property ψ next, i.e., $\varphi \subseteq T^{-1}(\psi)$. We can reformulate this as a Boolean algebra with operators (BAO): let A be the Boolean algebra of clopen subsets of X and let $\Box := T^{-1} : A \to A$. Then the Hoare triple is the conditional $a \to b := \neg a \vee \Box b$, which is valid when equal to X. So Stone duality not only links these BAOs to our dynamical domains but can also be regarded as a form of partial correctness.

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Local inconsistency lemmas and the inconsistency by cases property

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The study of deduction-detachment theorems and their algebraic counterparts is a classical part of abstract algebraic logic. It is a well-known fact that a finitary protoalgebraic logic has a deduction-detachment theorem – briefly a DDT – if and only if the semilattice of compact deductive filters of every algebra of the corresponding type is dually Brouwerian (see, e.g.,[4]). The bridge theorem has algebraic consequences, which in turn have logical applications crossing back over the bridge. For instance, any finitary protoalgebraic logic satisfying a DDT is filter-distributive, and the logical counterpart of filter-distributivity is the so-called *proof by cases property*, which has been extensively studied in [2, 3, 5].

In contrast, the theory of inconsistency lemmas, or ILs, for short, has not been systematically investigated so far, with a few exceptions (see, e.g., [1, 6, 7, 8]). Raftery proved in [8] that for a finitary protoalgebraic logic to have a (global) IL amounts to the demand that the join semilattice of compact deductive filters in each algebra of the corresponding type should be dually pseudo-complemented. Subsequently, Lávička [6] introduced and studied the local and parametrized local versions in a similar fashion to the hierarchy of DDTs.

Following the terminology introduced in [6], a logic $\vdash$ is said to have a *local inconsistency lemma*–briefly a LIL–if for every $n \in \mathbb{N}$, there exists a family Ψ_n of finite sets of formulas $I(x_1, \ldots, x_n)$ such that for every $\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \subseteq Fm$,

$$\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \text{ is inconsistent in } \vdash \iff \Gamma \vdash I(\varphi_1, \ldots, \varphi_n) \text{ for some } I \in \Psi_n.$$

The corresponding algebraic counterpart is the *maximal consistent filter extension property*, or MCFEP, for short, which a logic $\vdash$ is said to have if for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and every submatrix $\langle \boldsymbol{B}, G\rangle$ of $\langle \boldsymbol{A}, F\rangle$, for every maximal $\vdash$-filter H containing G there is a $\vdash$-filter H' containing F such that $H = H' \cap B$. This result established in [6] for protonegational logics [1] translates in the framework of finitary protoalgebraic logics as the following theorem:

Theorem 1. *Let $\vdash$ be a finitary protoalgebraic logic. The following are equivalent:*

1. *$\vdash$ has the LIL;*
2. *$\vdash$ has the MCFEP and for every algebra $\boldsymbol{A}$ the deductive filter A is finitely generated;*
3. *The MCFEP holds in the algebra of formulas and $\vdash$ posseses a finite inconsistent set of formulas.*

If the family Ψ_n witnessing the LIL consists of just one set of formulas $I(x_1, \ldots, x_n)$ for each $n \in \mathbb{N}$, then $\vdash$ is said to have an IL. As a first step to determine what is necessary for a LIL to reduce to an IL, we introduce the notion of *definable maximal consistent filters* – briefly DMCF. A logic $\vdash$ has DMCF if there is a formula $\delta(x_1, \ldots, x_n)$ in the language of the first-order

[1] The class of protonegational logics is introduced in [6] as a weakening of protoalgebraicity, restricting some of its defining conditions to maximal consistent theories. Particular examples are the negation fragments of protoalgebraic logics.

predicate logic (with equality), whose only non-logical symbols are the operation symbols of $\vdash$ and a unary predicate $F(x)$, such that for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and elements $a_1, \ldots, a_n \in A$,

$$A = \mathrm{Fg}_{\vdash}^{\boldsymbol{A}}(F \cup \{a_1, \ldots, a_n\}) \iff \langle \boldsymbol{A}, F\rangle \models \delta(a_1, \ldots, a_n).$$

In this case, for a finitary protoalgebraic logic $\vdash$ with an LIL and DMCF, we prove that any family Ψ_n witnessing the LIL must include a finite subset of sets of formulas for each $n \in \mathbb{N}$ such that the resulting family also witnesses the LIL for $\vdash$. However, the question of whether Ψ_n can be taken to be a singleton for every n, and obtain a global IL, is more involved.

Before answering this question, we introduce another notion: a logic $\vdash$ has the *inconsistency by cases property* (ICP) when for every nonnegative integers n, m, there exists a parameterized set $\nabla(x_1, \ldots, x_n, y_1, \ldots, y_m, \vec{z})$ of formulas such that for any set $\Gamma \cup \{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m\}$ of formulas, $\vec{\varphi} \vdash \vec{\varphi}\nabla\vec{\psi}$ and $\vec{\psi} \vdash \vec{\varphi}\nabla\vec{\psi}$, and whenever $\Gamma \cup \{\vec{\varphi}\}$ and $\Gamma \cup \{\vec{\psi}\}$ are inconsistent in $\vdash$, then $\Gamma \cup \{\vec{\varphi}\nabla\vec{\psi}\}$ is inconsistent in $\vdash$, where, $\vec{\varphi}\nabla\vec{\psi}$ is defined as $\bigcup\{\nabla(\vec{\varphi}, \vec{\psi}, \vec{\gamma}) : \vec{\gamma} \in Fm\}$.

It turns out that, in parallel to the connection between the proof by cases property and filter-distributivity, the corresponding bridge theorem arises between the ICP and the notion of 1-distributivity. Recall that a lattice $\boldsymbol{A}$ with 1 is said to be *1-distributive* if whenever $a \vee b = 1$ and $a \vee c = 1$, then $a \vee (b \wedge c) = 1$ for all elements $a, b, c \in A$. We obtain the following result:

Theorem 2. *Let $\vdash$ be a finitary protoalgebraic logic. The following are equivalent:*

1. *$\vdash$ has the ICP and possesses a finite inconsistent set of formulas;*
2. *For every algebra $\boldsymbol{A}$, the lattice of $\vdash$-filters of $\boldsymbol{A}$ is 1-distributive;*
3. *The lattice of theories of $\vdash$ is 1-distributive.*

Since every dually pseudo-complemented join semilattice with 1 is 1-distributive and any algebraic lattice is isomorphic to the lattice of ideals of the join semilattice of its compact elements, crossing back over the bridge to the syntactical setting, this implies that any finitary protoalgebraic logic with an IL has the ICP. Moreover, we prove that for a finitary protoalgebraic logic having a LIL witnessed by Ψ_n, the demand for the family to be directed for each $n \in \mathbb{N}$ amounts to the 1-distributivity of the logic. Consequently, a finitary protoalgebraic logic has an IL if and only if it has the MCFEP, for every algebra $\boldsymbol{A}$ the deductive filter A is finitely generated, it has DMCF and it is filter-1-distributive.

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A Discussion on Double Boolean Algebras
Extended Abstract

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In lattice theory, a *polarity* [1] is triple $\mathbb{K} := (G, M, I)$ where G and M are sets and $I \subseteq G \times M$. For any $X \subseteq G$, X^* is the set of all $m \in M$ such that gIm for all $g \in X$. For any $Y \subseteq M$, Y^+ is the set of all $g \in G$ such that gIm for all $m \in Y$. In formal concept analysis, a polarity is called a *context*. A *concept* is a pair of sets (X, Y) such that $X^* = Y$ and $X = Y^+$. The set of all concepts is denoted as $\mathcal{B}(\mathbb{K})$ and forms a complete lattice $\underline{\mathcal{B}}(\mathbb{K})$. The notion of a concept is generalized to protoconcepts and semiconcepts [4]. A *protoconcept* is a pair of sets (X, Y) such that $X^{*+} = Y^+$. A *semiconcept* is a pair of sets (X, Y) such that $X^* = Y$ or $X = Y^+$. We denote the sets of all protoconcepts and semiconcepts by $\mathcal{P}(\mathbb{K})$ and $\mathcal{H}(\mathbb{K})$, respectively. It is a straightforward observation that $\mathcal{B}(\mathbb{K}) \subseteq \mathcal{H}(\mathbb{K}) \subseteq \mathcal{P}(\mathbb{K})$. The meet ($\sqcap$) and join ($\sqcup$) operations of the complete lattice $\underline{\mathcal{B}}(\mathbb{K})$ are extended to the set of protoconcepts. Two negation operators $\neg$ and $\lrcorner$ are defined on the set $\mathcal{P}(\mathbb{K})$. With respect to the meet, join, and two negations, the set $\mathcal{P}(\mathbb{K})$ forms an algebraic structure which is called the *algebra of protoconcept*. The set of all semiconcept $\mathcal{H}(\mathbb{K})$ forms a subalgebra of the algebra of protoconcept and the subalgebra is called the *algebra of semiconcept*.

On the abstraction of the algebra of protoconcept and algebra of semiconcept, the definition of *double Boolean algebra* and *pure double Boolean algebra* are introduced. The definition of double Boolean algebra is given below.

Definition 1. [4] An algebra $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ satisfying the following properties is called a *double Boolean algebra* (dBa). For any $x, y, z \in D$,

$(1a) (x \sqcap x) \sqcap y = x \sqcap y$	$(1b) (x \sqcup x) \sqcup y = x \sqcup y$
$(2a) x \sqcap y = y \sqcap x$	$(2b) x \sqcup y = y \sqcup x$
$(3a) \neg(x \sqcap x) = \neg x$	$(3b) \lrcorner(x \sqcup x) = \lrcorner x$
$(4a) x \sqcap (x \sqcup y) = x \sqcap x$	$(4b) x \sqcup (x \sqcap y) = x \sqcup x$
$(5a) x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$	$(5b) x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$
$(6a) x \sqcap (x \vee y) = x \sqcap x$	$(6b) x \sqcup (x \wedge y) = x \sqcup x$
$(7a) \neg\neg(x \sqcap y) = x \sqcap y$	$(7b) \lrcorner\lrcorner(x \sqcup y) = x \sqcup y$
$(8a) x \sqcap \neg x = \bot$	$(8b) x \sqcup \lrcorner x = \top$
$(9a) \neg\top = \bot$	$(9b) \lrcorner\bot = \top$
$(10a) x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$	$(10b) x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
$(11a) \neg\bot = \top \sqcap \top$	$(11b) \lrcorner\top = \bot \sqcup \bot$
$(12) (x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$	

where $x \vee y := \neg(\neg x \sqcap \neg y)$ and $x \wedge y := \lrcorner(\lrcorner x \sqcup \lrcorner y)$. A quasi-order (that is reflexive and transitive) relation $\sqsubseteq$ on D is obtained as: $x \sqsubseteq y \iff x \sqcap y = x \sqcap x$ and $x \sqcup y = y \sqcup y$, for any $x, y \in D$.

Now we consider the two sets $D_\sqcap := \{x \in D : x \sqcap x = x\}$ and $D_\sqcup := \{x \in D : x \sqcup x = x\}$. A *pure* double Boolean algebra is a dBa $\mathbf{D}$ such that for $x \in D$, either $x \in D_\sqcap$ or $x \in D_\sqcup$. A dBa $\mathbf{D}$ is called *contextual* if the quasi-order becomes partial-order. Moreover, if for each $y \in D_\sqcap$ and $x \in D_\sqcup$ with $y \sqcup y = x \sqcap x$, there is a unique $z \in D$ with $z \sqcap z = y$ and $z \sqcup z = x$, $\mathbf{D}$ is called *fully contextual*.

This new algebraic structure opens up several possible research directions. In [3], we study the topological representation theorem for fully contextual dBa and pure dBa. The definition of double Boolean algebra contains a large number of axioms. However, we show that the axioms $(10a)$, $(10b)$, $(11a)$, and $(11b)$ are derivable from the remaining ones.

Theorem 1. Let $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ be an algebraic structure satisfying $(1a)-(9a)$, $(1b)-(9b)$, and 12 of Definition 1, then for all $x, y, z \in D$ the following hold.

(a) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$ and $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$.

(b) $\neg\bot = \top \sqcap \top$ and $\lrcorner\top = \bot \sqcup \bot$.

As the name suggests, for each dBa $\mathbf{D}$, there are two underlying Boolean algebras, $\mathbf{D}_\sqcap := (D_\sqcap, \sqcap, \neg, \bot)$ and $\mathbf{D}_\sqcup := (D_\sqcup, \sqcup, \lrcorner, \top)$. Moreover, the map $r : D \to D_\sqcap,\ r(x) := x \sqcap x$ preserves $\sqcap, \neg$ and $\bot$. The map $r' : D \to D_\sqcup,\ r'(x) := x \sqcup x$ preserves $\sqcup, \lrcorner$ and $\top$. We also have two injections $e : D_\sqcap \to D, e(x) = x$ and $e' : D_\sqcup \to D, e'(x) = x$ such that $r \circ e = id_{D_\sqcap}$ and $r' \circ e' = id_{D_\sqcup}$. Therefore, for a given dBa $\mathbf{D}$, we have the following:

(a) the semigroup $(D, \sqcap, \neg, \bot)$ satisfying $(1a)-(3a), (5a)-(8a)$, $(10a)$, and 12 is a retract [2] of the Boolean algebra $\mathbf{D}_\sqcap$.

(b) the semigroup $(D, \sqcup, \lrcorner, \top)$ satisfying $(1b)-(3b), (5b)-(8b)$, $(10b)$, and 12 is a retract [2] of the Boolean algebra $\mathbf{D}_\sqcup$.

The above observation gives the following representation theorem for dBa. We will sketch its proof in the talk.

Theorem 2. Let $(B, \wedge, \neg, \bot)$ and $(B', \vee', \neg', \top')$ be two Boolean algebras. Let $r : A \rightleftharpoons B : e$ and $r' : A \rightleftharpoons B' : e'$ be two embedding-retraction pair. $\mathbf{A} := (A, \sqcap, \sqcup, \neg, \lrcorner, e'(\top'), e(\bot))$ is a universal algebra where, $x \sqcap y := e(r(x) \wedge r(y)),\ x \sqcup y := e'(r'(x) \vee r'(y)),\ \neg x := e(\neg r(x))$, and $\lrcorner x := e'(\neg' r'(x))$. Then $\mathbf{A}$ is a dBa if and only if following holds.

(a) $e \circ r \circ e' \circ r' = e' \circ r' \circ e \circ r$.

(b) $e(r(x) \wedge r(e'(r'(x) \vee r'(y)))) = e(r(x))$ and $e'(r'(x) \vee r'(e(r(x) \wedge r(y)))) = e'(r'(x))$ for all $x, y \in A$.

(c) $r(e'(\top')) = \top$ and $r'(e(\top)) = \top'$.

Moreover, every dBa can be obtained from such an embedding-retraction construction.

In [4], it is shown that $D_p := D_\sqcup \cup D_\sqcap$ forms the largest pure subalgebra $\mathbf{D}_p$ of a dBa $\mathbf{D}$. Moreover, the largest pure subalgebra plays an important role in characterizing two different dBa. In particular, we will discuss the following result.

Theorem 3. Let $\mathbf{D}$ and $\mathbf{M}$ be fully contextual dBas. Then $\mathbf{D}$ is isomorphic to $\mathbf{M}$ if and only if $\mathbf{D}_p$ is isomorphic to $\mathbf{M}_p$. Moreover, every dBa isomorphism from $\mathbf{D}_p$ to $\mathbf{M}_p$ can be uniquely extended to a dBa isomorphism from $\mathbf{D}$ to $\mathbf{M}$.

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Approaching Rough Set Theory via Categories

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Pawlak's Rough Set Theory (briefly RST) is an elegant and powerful methodology, with applications in numerous research fields, aimed at the extraction and the optimization of the information coming from large amounts of data [9]. RST arose in the context of Pawlak's data tables with the purpose of understanding whether a given subset of objects could be partially or completely determined only on the basis of the information induced by collections of attributes.

The originary approach to RST is said *constructive*: to a suitable kind of relation $\mathcal{R}$ on the set U of the objects of a data table, one assigns a pair $(\mathrm{Lw}_{\mathcal{R}}, \mathrm{Up}_{\mathcal{R}})$ of dual set operators, respectively called $\mathcal{R}$-*lower* and $\mathcal{R}$-*upper approximants*, playing a similar role as the necessity and possibility operators in modal logic [7] and inducing a *constructive set algebra* $(\mathcal{P}(U), \cup, \cap, {}^c, \mathrm{Lw}_{\mathcal{R}}, \mathrm{Up}_{\mathcal{R}})$ on $\mathcal{P}(U)$, that often satisfies different algebraic properties [3, 6]. A further approach is said *algebraic*: one begins with a pair $(\mathbf{L}, \mathbf{H})$ of dual unary set operators defined axiomatically on a ground set U and, next, studies the resulting set algebra $(\mathcal{P}(U), \cup, \cap, {}^c, \mathbf{L}, \mathbf{H})$.

The common way of relating the previous two approaches comes from the characterization of the properties needed for defining an assignment $(\mathbf{L}, \mathbf{H}) \mapsto \mathcal{R}_{\mathbf{L},\mathbf{H}}$, where $\mathcal{R}_{\mathbf{L},\mathbf{H}}$ is a binary relation on U with $\mathbf{L}$ and $\mathbf{H}$ as lower and upper approximants [10]. Evidently, by adding suitable axioms in the definition of $\mathbf{L}$ and $\mathbf{H}$ we get additional properties on $\mathcal{R}_{\mathbf{L},\mathbf{H}}$. The attempts of combining the two approaches fit within the *representation problem*, aimed at the determination of those axiomatizations of $\mathbf{L}$ and $\mathbf{H}$ whose corresponding set algebra turns out to be the constructive set algebra induced by some specific kind of binary relation. For instance, by abstracting the axiomatic properties of $\mathrm{Lw}_{\mathcal{R}}$ and $\mathrm{Up}_{\mathcal{R}}$ when $\mathcal{R}$ is a *Pawlak's indiscernibility relation* [9], we get the so-called *lower* and *upper operators* and, in such a case, it follows that $\mathcal{R}_{\mathbf{L},\mathbf{H}}$ is an equivalence relation, yielding a *cryptomorphism* between all these structures [4].

The previous setting admits a natural categorical-theoretic interpretation - whose development might be useul to provide a unifying framework to RST - as soon as one asks questions about the functoriality of the assignments $\mathcal{R} \mapsto (\mathrm{Lw}_{\mathcal{R}}, \mathrm{Up}_{\mathcal{R}})$ and $(\mathbf{L}, \mathbf{H}) \mapsto \mathcal{R}_{\mathbf{L},\mathbf{H}}$ and of all the other constructions arising when developing the theory. To this end, the first necessity that has arisen concerns the definition of suitable categories to work with. Taking some ideas from the theory of combinatorial species, we use presheaves on the groupoid of sets to get a unifying framework in which to define collections of categories of mathematical structures with objects $\Omega_{\mathcal{X}}$, where Ω is an arbitrary set and $\mathcal{X}$ is either a specific n-ary relation, set operator, set system or data table with Ω as its attribute set.

However, in this context, the choice of the morphisms is not uniquely determined: for instance, when dealing with equivalence relations we can assume that the morphisms should preserve lower or upper approximants as required in [1, 8], or that the morphisms just preserve the equivalences as for the category **EqR**. The possibility of choosing the morphisms in completely different ways turns out to be fundamental in the attempt of making functorial various constructions on objects: as an example, some natural transformations among the presheaves that define the ambient categories within which to select the needed structures may be used to construct a non-trivial chain of categorical isomorphisms and embeddings involving suitable categories of equivalence relations, set partitions, upper and lower operators, in such a way to get a categorical counterpart for the cryptomorphism between these structures [4]. The previous result allows us to transport categorical properties from one category to another. To this end, to study the category of lower operators we made use of a particular category of set partitions, that has pullbacks and is regular, though it does not admits basic constructions as products, coproducts and coequalizers. In the same spirit, we made use of **EqR** to get informations on a category of upper operators and *continuos* functions: **EqR** becomes a specific case of *proper Moore-subcategory* of the category **Rel** of binary relations and relation-preserving maps. The investigation of proper Moore-subcategories of a given concrete category led to general results [5], holding for **EqR**: it is a reflective modification of **Rel** and inherits its limits and co-limits; it is **Set**-topological and **Set**-solid, has extremal subobject classifier but it is not regular [4].

Finally, to enrich our categorical framework for RST, after comparing possible definitions [2, 4] we introduce a category **PR** of Pawlak's data tables, obtained by dropping out the finiteness condition on its ground objects in view of possible theoretical applications from algebra and topology and assuming a compatibility condition on morphisms with interesting interpretations in applied contexts. There are at least three convincing reasons for working with **PR**. First, we proved that it is complete, balanced, exact, regular, Heyting, it admits (RegEpi, Mono-Source)-factorizations but, in general, not coproducts [4]. Secondly, being inspired by the existence of an embedding of **EqR** into **PR** that formalizes the fact that different subsets of attributes may induce the same Pawlak's indiscernibility, we can easily define convenient subcategories and functors through which to reinterpret in our categorical-theoretic setting various constructions of RST such as functional dependence or attribute reduction [4]. Third, **PR** becomes a specific instance of a further mathematical generalization, susceptible of an advanced study, by replacing sets with objects of an arbitrary category equipped with a symmetric monoidal structure.

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A comonadic account of Feferman–Vaught–Mostowski theorems

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Mostowski [5] showed that, for two pairs of logically equivalent relational structures $A \equiv_{FO} B$ and $A' \equiv_{FO} B'$ in first-order logic, their cartesian products are also logically equivalent $A \times A' \equiv_{FO} B \times B'$. Then, Feferman and Vaught [2] showed that a similar statement holds for arbitrary (potentially infinite) products and coproducts, instead of just binary products. In our work we give a categorical account of these and other Feferman–Vaught–Mostowski type theorems.

Aiming for applications in finite model theory, we reformulate these theorems in the recently introduced setting of *game comonads*. Typically, for a well-behaved fragment $\mathcal{L}$ of first-order logic, there is a comonad $\mathbb{C}$ on the category $\mathcal{R}(\sigma)$ of relational structures in signature σ. It is a standard fact about comonads that we have a pair of adjunctions and a comparison functor

$$\begin{tikzcd}
\mathsf{Kl}(\mathbb{C}) \arrow[rr, bend left, "K_{\mathbb{C}}"] \arrow[dr, bend right, "U_{\mathbb{C}}"'] & & \mathsf{EM}(\mathbb{C}) \arrow[dl, bend right, "U^{\mathbb{C}}"'] \\
& \mathcal{R}(\sigma) \arrow[ul, bend right, "F_{\mathbb{C}}"'] \arrow[ur, bend right, "F^{\mathbb{C}}"'] &
\end{tikzcd}$$

where $\mathsf{Kl}(\mathbb{C})$ is the Kleisli category and $\mathsf{EM}(\mathbb{C})$ is the Eilenberg–Moore category of coalgebras for $\mathbb{C}$. For our typical comonads, the objects of $\mathsf{EM}(\mathbb{C})$ can be viewed as tree-ordered relational structures and, moreover,

$$A \equiv_{\mathcal{L}} B \quad \iff \quad F^{\mathbb{C}}(A) \sim F^{\mathbb{C}}(B)$$

where $\sim$ denotes that the two structures in $\mathsf{EM}(\mathbb{C})$ are bisimilar. In fact, this bisimulation relation encodes that Duplicator/Player II has a winning strategy in the corresponding model comparison game for $\equiv_{\mathcal{L}}$. The structure of $F^{\mathbb{C}}(A)$ encodes all possible positions in this game. See [1] for a recent survey on game comonads.

Coming back to the theorem of Mostowski, we have a functor $\times \colon \mathcal{R}(\sigma) \times \mathcal{R}(\sigma) \to \mathcal{R}(\sigma)$ and we want to show that if

$$F^{\mathbb{C}}(A) \sim F^{\mathbb{C}}(B) \text{ and } F^{\mathbb{C}}(A') \sim F^{\mathbb{C}}(B') \quad \text{then also} \quad F^{\mathbb{C}}(A \times A') \sim F^{\mathbb{C}}(B \times B').$$

This indicates that we need to find a functor $\widetilde{\times} : \mathsf{EM}(\mathbb{C}) \times \mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{C})$ which commutes with the free functors $F^{\mathbb{C}}$ and preserves the bisimulation relation $\sim$.

A suitable candidate $\widetilde{\times}$ can be found by making use of the universal property of products. However, the task becomes more interesting when we abstract away from products and allow operations of arbitrary arity. It turns out that it is more natural to consider only unary operations between possibly different categories. This subsumes the n-ary case since the pointwise

*Joint work with Dan Marsden and Nihil Shah, mostly based on [4].

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product of n comonads on the product category is a comonad as well. To this end, assume that we have a functor $H\colon \mathcal{A} \to \mathcal{B}$ and comonads $\mathbb{C}$ and $\mathbb{D}$ on $\mathcal{A}$ and $\mathcal{B}$, respectively. As before, in order to have that

$$F^{\mathbb{C}}(A) \sim F^{\mathbb{C}}(B) \quad \text{implies} \quad F^{\mathbb{D}}(H(A)) \sim F^{\mathbb{D}}(H(B))$$

we need to find a *lifting* of H, that is, a functor $\widetilde{H}\colon \mathsf{EM}(\mathbb{C}) \to \mathsf{EM}(\mathbb{D})$ which preserves bisimulation and commutes with free functors: $F^{\mathbb{D}}(H(A)) \cong \widetilde{H}(F^{\mathbb{C}}(A))$. Observe that comonad morphisms $H\mathbb{C} \Rightarrow \mathbb{D}H$ are not suitable because the lift of H that these induce only commutes with the forgetful functors.

We take inspiration from the theory of monoidal monads (cf. [3, 6]), where the monoidal structure on the base category is lifted to the category of algebras for the monad. Perhaps surprisingly, the monoidal structure plays no role for the lift to exist. By dualising and generalising these results to our situation, we only require a Kleisli law $\mathbb{D}H \Rightarrow H\mathbb{C}$ (also known as an oplax comonad morphism) and $\mathsf{EM}(\mathbb{D})$ with equalisers of coreflexive pairs (ECP). Then the usual Kleisli lift $\widehat{H}$ of H further lifts to the categories of coalgebras:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F_{\mathbb{C}}"] \arrow[d, "H"'] & \mathsf{Kl}(\mathbb{C}) \arrow[r, "K_{\mathbb{C}}"] \arrow[d, "\widehat{H}"] & \mathsf{EM}(\mathbb{C}) \arrow[d, "\widetilde{H}"] \\
\mathcal{B} \arrow[r, "F_{\mathbb{D}}"] & \mathsf{Kl}(\mathbb{D}) \arrow[r, "K_{\mathbb{D}}"] & \mathsf{EM}(\mathbb{D})
\end{tikzcd}$$

Another surprising feature is that the theorems of [6] about bimorphisms generalise to this setting as well. These become crucial when proving that the lifted functor $\widetilde{H}$ preserves the bisimulation relation. In fact, our conditions ensure that $\widetilde{H}$ is a parametric relative right adjoint. To summarise, we prove the following.

Theorem 1. *Let $\mathbb{C}$ and $\mathbb{D}$ be comonads on $\mathcal{R}(\sigma)$ and $\mathcal{R}(\tau)$, capturing logic fragments $\mathcal{L}$ and $\mathcal{K}$, respectively. Assume $\mathsf{EM}(\mathbb{D})$ has ECP and $H\colon \mathcal{R}(\sigma) \to \mathcal{R}(\tau)$ admits a Kleisli law $\mathbb{D}H \Rightarrow H\mathbb{C}$ which lifts H to a parametric relative right adjoint between the categories of coalgebras then*

$$A \equiv_{\mathcal{L}} B \quad \textit{implies} \quad H(A) \equiv_{\mathcal{K}} H(B).$$

Not only many Feferman–Vaught–Mostowski type theorems from the literature are a special case of this theorem but, also, this theorem becomes essential in the theory of game comonads. It allows us to compare logics, show preservation of type-equivalence by transformations, prove locality theorems, etc.

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On (modal) expansions of pointed Abelian logic

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The variety of lattice-ordered Abelian groups (Abelian ℓ-groups, for short) is well known and studied [5]. It was established in [7] that, as a quasivariety, it is generated by the ℓ-group of integer numbers. Abelian ℓ-groups are not only interesting from the algebraic point of view but also from the logical point of view, since they form the algebraic semantics for Abelian logic (see [1, 8]). Since the quasivariety of Abelian ℓ-groups have no proper subquasivarieties, Abelian logic has no proper extensions by finitary rules.

Another logical motivation for the study of Abelian ℓ-groups is the key role they play in understanding Łukasiewicz logic; see for example the proof of Chang's theorem [2]. In general, the study of Abelian ℓ-groups is very closely related to the study of MV-algebras, since these two classes of structures are connected via Mundici functor [2]. It is also worth mentioning that Abelian logic can be seen in as a weakening-free variant of Łukasiewicz logic.

In this talk, we will discuss the class of pointed Abelian ℓ-groups and its corresponding logic, which we call pointed Abelian logic. By a pointed Abelian ℓ-group we mean an Abelian ℓ-group with one additional fixed element in the signature without any additional property. While it may seem that it is not more than a cosmetic change, it turns out that the additional constant symbol allows us to express many new no-trivial logical axiom/rules and so the corresponding lattice of sub(quasi)varieties of pointed Abelian ℓ-groups is quite complex and worth exploring.

In previous research we have discussed several extensions of pointed Abelian logic, the most important of which was a finitary version of the unbounded Łukasiewicz logic. This logic was introduced (but not named) in [3], along with its philosophical and linguistic motivation, and has a clear mathematical motivation: it combines Abelian logic and Łukasiewicz logic in a very natural way.

In this talk, we focus on an infinitary version of the unbounded Łukasiewicz logic, i.e., a logic strongly complete with respect to the pointed ℓ-group of reals with point at -1.

To this end, we must first prove some general results about extensions of (pointed) Abelian logic using infinitary rules (recall here that we cannot get any non-trivial extension of Abelian logic by using additional finitary rules). We will give axiomatizations of some of these extensions and show that there are uncountably many of them. In particular, we focus on those whose corresponding algebraic semantics is the generalized subquasivariety generated by Archimedean ℓ-groups, integers and reals (in pointed case with positive or negative interpretation of the point).

We conclude the talk by exploring the addition of modalities to (the extensions of) pointed Abelian logic. We follow the footsteps of [4] (for Abelian logic) and [4, 6] (for Łukasiewicz logic) and focus on the differences caused by the presence of the additional constant and the lack of weakening.

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Finite lattices, N-free posets and orthomodularity

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Dacey orthosets

In his PhD. thesis [1], Dacey explored the notion of "abstract orthogonality" through sets equipped with a symmetric, irreflexive relation $\perp$. He named these structures *orthogonality spaces*. More recently, these structures have been referred to as *orthosets* [6, 5] and we will adopt this terminology in this text. Note that an orthoset is just a simple graph.

Definition 1. An *orthoset* $(O, \perp)$ is a set O equipped with an irreflexive symmetric binary relation $\perp \subseteq O \times O$, called *orthogonality*.

For every subset X of an orthoset O, we write

$$X^{\perp} = \{y \in O \mid \text{for all } x \in X,\ x \perp y\}$$

Graph-theoretically, this is just the set of all common neighbours of a set of vertices. It is easy to see that $X \mapsto X^{\perp\perp}$ is a closure operator; a subset of an orthoset $(O, \perp)$ with $X = X^{\perp\perp}$ is called *orthoclosed*. The orthoclosed subsets of an orthoset O form a complete ortholattice $L(O, \perp)$, which we call *the logic of* $(O, \perp)$.

Arguably the most significant theorem established in Dacey's thesis [1] is the following.

Theorem 1. *Let $(O, \perp)$ be an orthoset. Then $L(O, \perp)$ is an orthomodular lattice if and only if for every orthoclosed subset X and every maximal pairwise orthogonal set (or a clique) $B \subseteq X$, $B^{\perp\perp} = X^{\perp\perp}$.*

The orthosets that meet one of the equivalent conditions of Theorem 1 are called *Dacey orthosets*.

The basic idea

Recently, we have achieved moderate success using the following straightforward approach.

1. We consider some class of (finite) objects $\mathcal{X}$.
2. For every object $X \in \mathcal{X}$, we construct in some way an orthoset $\mathcal{O}(X)$.
3. We characterize those objects $X \in \mathcal{X}$ for which $\mathcal{O}(X)$ is a Dacey orthoset.
4. We characterize those objects $X \in \mathcal{X}$ for which $\mathcal{O}(X)$ is a Dacey orthoset with a Boolean logic.

In the talk, we will present several results we found using this approach. The class $\mathcal{X}$ will be always some class of posets. We will consider two types of a construction of an orthoset from a poset: the orthoset of quotients and the incomparability orthoset.

Orthosets of quotients

For a poset P, we write $Q^{+}(P)$ for the set of all pairs $(a, b) \in P \times P$ with $a < b$. In lattice theory, the elements of $Q^{+}(A)$ are called *proper quotients*. An element $(a, b) \in Q^{+}(P)$ is denoted by $[a < b]$.

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For $[a < b], [c < d] \in Q^+(P)$ we write $[a < b] \perp [c < d]$ if $b \leq c$ or $d \leq a$. Clearly, $\perp$ is symmetric and irreflexive, so $(Q^+(P), \perp)$ is an orthoset.

Theorem 2. *[3] Let P be a finite bounded poset. Then P is a lattice if and only if its orthoset of quotients $(Q^+(P), \perp)$ is Dacey.*

Theorem 3. *[3] Let P be a bounded poset. Then P is a chain if and only if $L(Q^+(P), \perp)$ is a Boolean algebra.*

Incomparability orthosets

Let P be a poset. For $x, y \in P$ let us now write $x \perp y$ if and only if x, y are incomparable. We say that $(P, \perp)$ is the *incomparability orthoset* of P.

Let P be a poset. For a quadruple of elements $(a, b, c, d) \in P^4$, we say that they *form an N* if and only if $a < c \succ b < d$ (note the covering relation here), $b < d$, and all the other distinct pairs of elements of the set $\{a, b, c, d\}$ are incomparable. We denote this by $N(a, b, c, d)$. A poset such that no quadruple of elements forms an N is called *N-free*.

N-free posets were introduced by Grillet in [2]. In that paper, the following characterization of N-free posets was proved.

Theorem 4. *A finite poset P is N-free if and only if every maximal chain in P intersects every maximal antichain in P.*

Theorem 5. *[4] Let P be a finite poset. Then P is N-free if and only if its incomparability orthoset $(P, \perp)$ is Dacey.*

We further characterize the finite posets P with a Boolean $L(P, \perp)$ by the absence a more general type of small substructure, which we term a *weak N*; a weak N is like the N defined earlier, with the distinction that we allow a and d to be comparable.

Theorem 6. *[4] Let P be a finite poset. Then $L(P, \perp)$ is Boolean iff there is no weak N in P.*

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Semicartesian categories of relations

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Quantization is the process of generalizing mathematical structures to the noncommutative setting. Many quantum phenomena have classical counterparts, and can often be modelled by quantized versions of the mathematical structures modelling these classical counterparts. Recently, several mathematical structures have been quantized via a quantization method based on Weaver's notion of a *quantum relation* between von Neumann algebras [13], which he distilled from his work with Kuperberg on the quantization of metric spaces [12]. Quantum relations can be regarded as noncommutative versions of ordinary relations, and admit a rich relational calculus that allows us to generalize concepts such as symmetric, antisymmetric, reflexive, and transitive relations to the noncommutative setting. Building on these concepts, Weaver quantized posets [13] and showed that *quantum graphs* [2], which are used for quantum error correction, can be understood in terms of quantum relations [14].

Von Neumann algebras are noncommutative generalizations of measure spaces rather than of sets. Kornell identified *hereditarily atomic* von Neumann algebras, which are essentially (possibly infinite) sums of matrix algebras, as the proper noncommutative generalizations of sets [8]. For this reason, hereditarily atomic von Neumann algebras are also called *quantum sets*, and the category **qRel** of quantum sets and quantum relations can be regarded as the proper noncommutative generalization of the category **Rel** of sets and binary relations. Just like **Rel**, but in contrast to the category of all von Neumann algebras and quantum relations, **qRel** is dagger compact closed. Together with Kornell and Mislove, the second author investigated the categorical properties of quantum posets in this restricted setting of hereditarily atomic von Neumann algebras [11]. Building on this work, they introduced *quantum cpos*, which are noncommutative versions of ω-complete partial orders (cpos). Ordinary cpos can be used to construct denotational models of ordinary programming languages, and in a similar way, they showed that quantum cpos can be used for the denotational semantics of quantum programming languages [10]. Also building on the definition of quantum posets in the hereditarily atomic setting, both authors introduced *quantum suplattices* [7], which are noncommutative versions of complete lattices and supremum-preserving maps. For quantum suplattices, the compact structure of **qRel** seems to be essential.

Categorically, quantization via quantum relations can be understood as the internalization of mathematical structures in the category **qRel**, and many theorems about quantized structures via quantum relations rely on the categorical properties of **qRel**. There are several categorical generalizations of the category **Rel** such as allegories [3] or bicategories of relations [1], but unfortunately, **qRel** is not an example of either of them. This is mainly due to the fact that the internal functions of **qRel** form a semicartesian monoidal category rather than a cartesian monoidal category, which reflects the quantum character of **qRel**. Tweaking the definitions of either allegories or bicategories of relations is difficult; their cartesian character seems to be essential, and it cannot be adjusted without tearing down the whole building.

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Therefore, we aim to find a different categorical generalization of **Rel** that would capture **qRel**, but also other generalizations of **Rel**, such as the category V-**Rel** of sets and relations with values in a unital commutative quantale V, which is used in fuzzy mathematics [6]. Only if V is a frame, V-**Rel** seems to be a bicategory of relations. We take daggers as a primitive notion, and identify six properties of **qRel** as axioms for our categorical generalization of **Rel**. Similar properties also occur in recent categorical axiomatizations of several dagger categories such as the category **Hilb** and **Rel** [4, 9, 5], and likely will form a subset of the axioms of a future categorical characterization of **qRel**. Hence, we define a *semicartesian category of relations* to be a category $\mathbf{R}$ such that

(1) $\mathbf{R}$ is a locally small dagger compact category;

(2) $\mathbf{R}$ has all small dagger biproducts;

(3) $\mathbf{R}$ has precisely two scalars;

(4) $\mathbf{R}$ is a dagger kernel category;

(5) For each object X in $\mathbf{R}$ there is precisely one morphism $X \to I$ with zero kernel;

(6) For each object X and each projection p on X, $p \geq \mathrm{id}_X$ if and only if $\ker p = 0$.

Here, a *projection* on an object X is a morphism $p : X \to X$ such that $p \circ p = p = p^\dagger$. For the last axiom, we use that the first three axioms imply that $\mathbf{R}$ is a *quantaloid*, i.e., a category enriched over the category **Sup** of complete lattices and supremum-preserving maps. As another consequence of the axioms, we prove that the homsets of $\mathbf{R}$ are actually orthomodular lattices. We conclude with a discussion of conditions that assure the existence of a power set construction in semicartesian categories of relations.

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A Categorical Equivalence for Odd or Even Involutive FL_e-chains

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The main objective of this talk is to lift the established one-to-one correspondence, as introduced in [6], between the class of even or odd involutive FL_e-chains and the class of bunches of layer groups, to a categorical equivalence [4]. In [6] a novel decomposition method, called *layer algebra decomposition*, which seems to be original not only in the field of residuated lattices but also in algebra at large (see footnote 4 in [6]), along with the corresponding construction method, have been introduced for the class of odd or even involutive FL_e-chains. The main idea was to decompose the algebra with the help of its local unit function $x \mapsto x \to x$ into a direct system, indexed by the positive idempotent elements of the algebra, of (hopefully simpler, "nicer") algebras, with transitions of the direct system defined by multiplication by a positive idempotent element. The decomposed algebra could be reconstructed through a combination of Płonka sums, as introduced by Płonka [9], and the concept of directed lexicographic order, introduced in [6] (see Remark 4.3). The impact of the layer algebra decomposition soon extended beyond its initial application and has been employed to structurally describe various classes of residuated lattices. These include finite commutative, idempotent, and involutive residuated lattices [8], finite involutive po-semilattices [7], and locally integral involutive po-monoids and semirings [1, 2]. In these classes layer algebras are "nice". However, in [6] the obtained layer algebras are only somewhat nicer than the original algebra, therefore a second phase, involving the construction of layer groups from layer algebras, was introduced. The combination of the layer algebra decomposition and this additional phase establishes a one-to-one correspondence between the class of even or odd involutive FL_e-chains and the class of bunches of layer groups. With the obvious choice for morphisms between FL_e-chains, our primary focus in this talk is to determine the appropriate notion of morphisms for the class of bunches of layer groups, and to present a functor. Due to space constraints here, we direct the interested reader to [4, Definition 2.3 and Remark 2.6] for the description of bunches of layer groups (the objects of the category of bunches of layer groups), to [4, Definition 3.3] for the description of bunch homomorphisms (the morphisms of the category of bunches of layer groups), and to [4, Theorem 3.6] for an explanation of a functor mapping to the category of even or odd involutive FL_e-chains. As a forward-looking note, it's worth noting that the categorical equivalence presented in this talk (and in [4]), has proven to be a potent tool for establishing amalgamation and densification results in classes of involutive FL_e-algebras that are neither integral, nor divisible, nor idempotent [3, 5].

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On the structure of balanced residuated posets

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A residuated poset is a structure of the form $(A, \leq, \cdot, 1, \backslash, /)$ such that $(A, \leq)$ is a poset, $(A, \cdot, 1)$ is a monoid and $x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z$. It is called *balanced* if it satisfies the identity $x/x = x\backslash x$, or equivalently all positive idempotents are central (i.e., $1 \leq x = x^2 \implies x \cdot y = y \cdot x$). In this case we denote the term x/x by 1_x and call it a local identity since it satisfies $1_x \cdot x = x = x \cdot 1_x$.

We show that any balanced residuated poset can be decomposed into components $C_x = \{y : 1_y = 1_x\}$ and two families of maps from which the original residuated poset can be reconstructed. If the balanced residuated poset satisfies the identities $1_{x \cdot y} = 1_x \cdot 1_y = 1_{x/y} = 1_{x\backslash y}$ then it decomposes as a Płonka-style sum over a semilattice direct system of integral residuated posets. This structure theory generalizes the results in [1] where the residuated posets were assumed to be involutive and locally integral, hence square-decreasing.

The construction of Płonka sums from finite families of finite integral involutive residuated posets has been implemented in Python. To allow for a convenient specification of semilattice direct systems of maps, we define dual partial function systems over sets of indecomposable residuated posets. If the partial functions are assumed to be continuous with respect to Stone spaces on their domain and codomain then the components of the semilattice direct systems are Boolean products over these Stone spaces.

The glueing construction in [2] for finite commutative idempotent involutive residuated lattices produces lattice-ordered algebras rather than po-algebras. In the setting of involutive residuated lattices without finiteness, commutativity or idempotence, we show that the glueing of two integral components, over an isomorphic filter and ideal in the respective component, again produces an involutive residuated lattice. Ongoing research aims to extend this result to Płonka-style sums of balanced residuated lattices.

The results reported here are joint research with Stefano Bonzio, José Gil-Férez, Adam Přenosil and Melissa Sugimoto.

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Semi-divisible lattices and Modal operators

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The class of semi-divisible residuated lattices is a wide class of algebraic structures. In fact, Turunen [6] shows that all divisible residuated lattices (Rl-monoids) are semi-divisible residuated lattices but not all semi-divisible residuated lattice are Rl-monoids. So, divisible residuated lattices are subclass of semi-divisible residuated lattices. Turunen in [6], while studying states on semi-divisible residuated lattices proved that for some states, study them on semi-divisible residuated lattices or on divisible residuated lattices is the same. This proves that there are notions which, once studied in the divisible residuated lattices setting, there is no longer any point in moving on to the semi-divisible residuated lattices context.
Modal operators were first defined and studied on Heyting algebras in 1981 by Macnab [3]. Since then, many authors have investigated properties of modal operators on other classes of residuated lattices: Harlenderovand Rachunek [1] studied modal operators on MV-algebra, Rachunek and Salounov[4] studied modal operators on Rl-monoids, monotone modal operators on bounded integral residuated lattices were studied by Rachunek et al [5] and Kondo [2] studied modal operators on commutative residuated lattices. One of the goals behind the study of modal operators is to build special cases of closure operators which are important for the theoretical study of partial ordered sets. These special cases of closure operators are monotone modal operators. Rachunek and Salounov[4] proved when studying modal operators on Rl-monoids, that, all modal operators on divisible residuated lattices are closure operators. It becomes really interesting to know if this is the case in a context of semi-divisible residuated lattices.

We start this paper by showing that not all modal operators on semi-divisible residuated lattices are closure operators. We also investigate some properties of modal operators on a residuated lattice which are not found in literature and state some conditions for a modal operator to be a closure operator on a semi-divisible residuated lattice. Then, we define some operators on a semi-divisible residuated lattice and show that they are strong modal operators. We end the paper by constructing a semi-divisible residuated lattice which is not an Rl-monoid from an idempotent element of a semi-divisible residuated lattice and we define a monotone modal operator on it.

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Semi-prelinear Residuated Lattices

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The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull [11], Dilworth [4], Balbes and Dwinger [1] and Pavelka [12]. In [9], Idziak proved that the class of residuated lattices is equational. Several authors have introduced different sub-classes of residuated lattices such as De Morgan residuated lattices [8], quasicomplemented residuated lattices [13], divisible residuated lattices [2], semi-divisible residuated lattices [2], MTL-algebra [5], BL-algebra and G-algebra [6] to name only these.

During the last decades, the fuzzy logic has become very popular, mainly because of its applicational aspects. In the framework of "soft computing", continuous triangular t-norms are used as "conjunction" and the corresponding residuum as "implication" to combine fuzzy sets with membership values in $[0, 1]$. This has a great importance in applicational aspects, particularly in fuzzy control, uncertain modeling, graph theory, data visualization and analysis. BL-algebras have been invented by Hajek [6] in order to provide an algebraic proof of the completeness of basic fuzzy logic (BL for short), the logic of continuous t-norms and their residua. However, a sufficient and necessary condition for a t-norm to have a residuated implication is left-continuity; hence it makes sense to consider fuzzy logics based not on continuous t-norms but on left-continuous t-norms. To this end, Esteva and Godo proposed in [5] a new logic, called MTL, as the basic fuzzy logic in this more general sense. The proposal was successful when Jenei and Montagna proved in [10] that MTL is indeed the logic of all left-continuous t-norms and their residua. The algebraic models of MTLs are MTL-algebras, the divisibility condition $x \odot (x \to y) = x \wedge y$ does not hold, so they are residuated lattices verifying only the prelinearity condition $(x \to y) \vee (y \to x) = 1$. MTL-algebras have been widely studied in the literature [5, 3]. It has been proven that anintegral idempotent residuated lattice is a Heyting algebra, hence idempotency is a very strong notion in the residuated lattice setting. Various special residuated lattices are now used as the main structure of truth values in fuzzy set theory and are subject to algebraic investigation.

In the present paper, we introduce the concept of semi-idempotent residuated lattices, provide new characterizations and establish many of their important properties. In addition, introduce a new subclass of residuated lattices called semi-prelinear residuated lattices. This is done by replacing the prelinearity axiom $(x \to y) \vee (y \to x) = 1$ by a weaker axiom: $(x' \to y') \vee (y' \to x') = 1$ where $x' = x \to 0$, called semi-prelinearity equation. This study was motivated in part by the fact that a similar approach was used to treat the concept of semi-divisibility by weakening the divisibility axiom $x \odot (x \to y) = x \wedge y$ to: $[x' \odot (x' \to y')]' = (x' \wedge y')'$. We study semi-prelinear residuated lattices and establish the links with several subclasses of residuated lattices. Many results similar to those obtained for MTL-algebras are obtained. The different situations depicted with prime ideals and filters show the gap between semi-prelinearity and prelinearity. In order to stress the divide between the two classes of residuated lattices, we point at some important results that hold in prelinear settings but not in semi-prelinear ones

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(see., e.g., [16, Theorem 4]). Finally, we extend the work of Belohlavek and Vychodil [14] by computing the numbers of residuated lattices such as semi-prelinear, semi-divisible, De Morgan, Stonean and semi-idempotent up to order 12.

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Conjunctive Table Algebras

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Tables. The first infinite ordinal is denoted by ω. We formalize a *table* as a set $T \subseteq G^X$, where $X \subseteq \omega$ is a finite set of *column names* (not column numbers), an element $t \in T$ is a *row*, $t(x)$ is the *entry* in row t and column x, and G is an arbitrary set. Hence,

$$\mathrm{Tab}(G) = \bigcup\{\mathfrak{P}(G^X) \mid X \subseteq \omega \text{ finite}\} \tag{1}$$

contains all tables with entries in G. Note that while X must be finite, a table can have an infinite number of rows if G is infinite.

Primitive Positive Formulas. Let M denote a relational signature. A first-order formula over M is *primitive positive* if it is built from atoms using $\{\wedge, \exists\}$. An atom is either a *relational atom* $Rx_1 \dots x_n$, an *equality atom* $x{=}y$, or one of the special atoms true (the *tautology*) and false (the *contradiction*). The set of primitive positive formulas over M is denoted by $\mathrm{PP}(M)$.

Variables. We assume that ω is the countably infinite set of variables. The function free : $\mathrm{PP}(M) \to \mathfrak{P}(\omega)$ maps each formula φ to the set of free variables occurring in φ; for the special atoms, we define $\mathrm{free}(\mathsf{true}) = \emptyset$ and $\mathrm{free}(\mathsf{false}) = \omega$.

Conjunctive Table Algebras. Every relational structure $\mathfrak{G}$, with universe G and signature M, induces a *solution operation* $(\cdot)^{\mathfrak{G}} : \mathrm{PP}(M) \to \mathrm{Tab}(G)$ that maps each formula φ to its *solution set*

$$\varphi^{\mathfrak{G}} := \{t \in G^{\mathrm{free}(\varphi)} \mid \mathfrak{G} \models \varphi[t]\} \subseteq \mathrm{Tab}(G) \quad , \tag{2}$$

where $\mathfrak{G} \models \varphi[t]$ means that φ holds in $\mathfrak{G}$ under the variable assignment $t : \mathrm{free}(\varphi) \to G$.

The algebra $\mathbf{PP}(M) := (\mathrm{PP}(M), \wedge, \mathsf{false}, \mathsf{true}, \exists_x, x{=}y, \mathrm{free})_{x,y\in\omega}$ extends $\mathrm{PP}(M)$ with a binary operation $\wedge$ (interpreted as syntactic conjunction), a unary operation $\exists_x$ for each $x \in \omega$ (interpreted as syntactic existential quantification over x), the function $\mathrm{free} : \mathrm{PP}(M) \to \mathfrak{P}(\omega)$, and it contains all non-relational atoms as distinguished elements. The solution operation homomorphically maps the logical operations to corresponding table operations; we have

$$(\varphi \wedge \psi)^{\mathfrak{G}} = \varphi^{\mathfrak{G}} \bowtie \psi^{\mathfrak{G}}, \quad \mathsf{false}^{\mathfrak{G}} = \emptyset, \quad \mathsf{true}^{\mathfrak{G}} = \{\emptyset\}, \quad (\exists_x \varphi)^{\mathfrak{G}} = \mathrm{del}_x(\varphi^{\mathfrak{G}}), \quad (x = y)^{\mathfrak{G}} = E_{xy},$$

where $\bowtie$ is the *natural join*, $\emptyset$ is the *empty table*, $\{\emptyset\}$ is the table with a single empty row, del_x is a *deletion operation* (deletes column x if it exists), and $E_{xy} := \{t \in G^{\{x,y\}} \mid t(x) = t(y)\}$ is a *diagonal*. Moreover, the *schema* of a table $T \in \mathrm{Tab}(G)$ is uniquely defined by

$$\mathrm{schema}(T) := \begin{cases} X & \text{if } T \in G^X \text{ and } T \neq \emptyset \\ \omega & \text{if } T = \emptyset \end{cases} \quad , \tag{3}$$

and if $\varphi^{\mathfrak{G}} \neq \emptyset$, then also $\mathrm{free}(\varphi) = \mathrm{schema}(\varphi^{\mathfrak{G}})$. This motivates the definition of the table algebra $\mathbf{Tab}(G) := (\mathrm{Tab}(G), \bowtie, \emptyset, \{\emptyset\}, \mathrm{del}_x, E_{xy}, \mathrm{schema})_{x,y\in\omega}$. A *conjunctive table algebra* with *base* G is a subalgebra of $\mathbf{Tab}(G)$.

Comparison with cylindric set algebras. Conjunctive table algebras are a database-theoretic variant of cylindric set algebras (of dimension ω). In his survey paper [2, Sect. 7(4)], Németi briefly discusses the charm of such a variant. Németi's universe $\mathrm{Gfs}(G)$ is our $\mathrm{Tab}(G)$. He credits Howard [1] with the approach (although Howard refers to the universe $\mathfrak{P}(\bigcup_{X \subseteq \omega} G^X)$). Howard uses complements, so in that sense, conjunctive table algebras are more generic.

Main Result. We present an axiomatization of conjunctive table algebras. The conjunctive table algebras with nonempty base are, up to isomorphism, precisely the projectional semilattices; a *projectional semilattice* is an algebraic structure $(V, \wedge, 0, 1, c_x, d_{xy}, \mathrm{dom})_{x,y \in \omega}$ consisting of an infimum operation $\wedge$, a *bottom element* 0, a *top element* 1, a *cylindrification* $c_x : V \to V$ for each $x \in \omega$, a *diagonal* $d_{xy} \in V$ for each $(x, y) \in \omega \times \omega$, and a *domain function* $\mathrm{dom} : V \to \mathfrak{P}(\omega)$, such that the axioms

(PS0) $(V, \wedge, 0, 1)$ is a bounded semilattice

(PS1) $c_x(0) = 0$

(PS2) $u \leq c_x(u)$

(PS3) $c_x(u \wedge c_x(v)) = c_x(u) \wedge c_x(v)$

(PS4) $c_x(c_y(u)) = c_y(c_x(u))$

(PS5) $u \neq 0 \Rightarrow (u \neq c_x(u) \Leftrightarrow u \leq d_{xx})$

(PS6) $x \neq y, z \Rightarrow d_{yz} = c_x(d_{yx} \wedge d_{xz})$

(PS7) $x \neq y \Rightarrow d_{xy} \wedge c_x(d_{xy} \wedge u) \leq u$

(PS8) $u \neq 0 \Rightarrow \mathrm{dom}(u)$ finite

(PS9) $\mathrm{dom}(u) = \{x \in \omega \mid u \leq d_{xx}\}$

(PS10) $\mathrm{dom}(u) = \emptyset \Rightarrow u = 1$

(PS11) $d_{xx} \neq 0$

(PS12) $d_{xy} = d_{yx}$

hold for all $u, v \in V$ and $x, y, z \in \omega$.

Comparison with cylindric algebras. The axioms (**PS0**), . . . , (**PS7**) correspond to cylindric algebra axioms (**CA0**), . . . , (**CA7**). Axiom (**CA0**) asserts a Boolean algebra; since we do not consider disjunction and negation, axiom (**PS0**) only asserts a bounded semilattice. The Axioms (**CA1**), (**CA2**), (**CA3**), (**CA4**) and (**CA6**) are identical to (**PS1**), (**PS2**), (**PS3**), (**PS4**) and (**PS6**), respectively. Cylindric algebra axiom (**CA5**) states $d_{xx} = 1$, reflecting that $x{=}x$ is a tautology; however, the *table semantics* in eq. (2) corresponds to a logic with undefined variables, where $x{=}x$ is not a tautology! We consider (**PS5**) to be a suitable replacement: Under the definition axiom (**PS9**), axiom (**CA5**) asserts $\mathrm{dom}(u) = \omega$ for all $u \neq 0$; whereas axiom (**PS5**) asserts $\mathrm{dom}(u) = \{x \in \omega \mid c_x(u) \neq u\}$ for all $u \neq 0$; the latter set is known as the *dimension set* $\Delta(u)$ in the terminology of cylindric algebras. Axiom (**PS7**) is the historical axiom (**CA7**); the contemporary axiom (**CA7**) is equivalent but involves negation! Historically, there was also an axiom (**CA8**), stating that $\Delta(u)$ is finite for all $u \in V$. Since $\mathrm{dom}(u) = \Delta(u)$ for $u \neq 0$, we can identify (**CA8**) with (**PS8**), disregarding the case $u = 0$.

Variant: Empty Universe. If axiom (**PS11**) is weakened to $1 \neq 0$, we obtain a characterization of conjunctive table algebras (including base $G = \emptyset$) up to isomorphism.

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Implication free reduct of intuitionism, or p-algebras revisited

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Abstract

We give a new description of free (distributive) p-algebras, which in particular yields a normal form theorem for terms. We also prove some new results about the subquasivariety lattice, which for lack of space we can only signal below.

1 Introduction

A *distributive p-algebra* (from now on, simply, a *p-algebra*) is an algebra $(A; \wedge, \vee, ^*, 0, 1)$ where $(A; \wedge, \vee, 0, 1)$ is a bounded distributive lattice, and the unary operation * satisfies the equivalence

$$x \wedge y = 0 \iff x \leq y^*.$$

P-algebras are a variety, Pa, consisting of term-subreducts of Heyting algebras, without implication but with the term $x^* := x \to 0$. Their subvariety lattice is a chain of type $\omega + 1$:

$$\mathsf{Pa}_{-1} \subset \mathsf{Pa}_0 \subset \mathsf{Pa}_1 \subset \cdots \subset \mathsf{Pa}$$

where Pa_{-1} is the trivial variety, and Pa_k is generated by a single subdirectly irreducible algebra. In contrast to Heyting algebras, they are not 1-regular and not even 1-subtractive, although they are 0-subtractive (yet still not 0-regular).

They were studied extensively in 1970s and 1080s, and then the research petered out. We try to rekindle the interest in these algebras providing a new description of free p-algebras, based entirely on a thorough understanding of completely meet-irreducible congruences. This enables us to formulate a normal form theorem for p-algebra terms, and prove some results about the lattice of subqusivarieties of p-algebras, sharpening the existing ones.

2 Free algebras

We build free algebras using completely meet-irreducible congruences. The most important observation on them is that they come in two layers, given in the following definition, where $\mathbf{Cm}\,\mathbf{A}$ stands for the set of completely meet-irreducible congruences of $\mathbf{A}$; for $\mu \in \mathbf{Cm}\,\mathbf{A}$, we write μ^+ for the unique cover of μ in the lattice $\mathbf{Con}\,\mathbf{A}$ of all congruences of $\mathbf{A}$, and $M(\alpha)$ is $\{\mu \in \mathbf{Cm}\,\mathbf{A} : \alpha \subseteq \mu\}$.

Definition 2.1. *Let* $\mathbf{A} \in \mathsf{Pa}$. *Put*

$$\mathrm{I}_{\mathbf{A}} := \{\mu \in \mathbf{Cm}\,\mathbf{A} : \mathbf{A}/\mu \cong \overline{\mathbf{B}}_0\} = \{\mu \in \mathbf{Cm}\,\mathbf{A} : \mu^+ = \mathbf{1}^{\mathbf{A}}\},$$
$$\mathrm{II}_{\mathbf{A}} := \{\mu \in \mathbf{Cm}\,\mathbf{A} : \mathbf{A}/\mu \cong \overline{\mathbf{B}}_n \textit{ for } n > 0\} = \{\mu \in \mathbf{Cm}\,\mathbf{A} : M(\mu^+) \subseteq \mathrm{I}_{\mathbf{A}}\}.$$

Let $\mathbf{F}_n(k)$ be the free k-generated p-algebra in the variety Pa_n. Let $T \in \mathcal{P}(k)$, define $f_T : \{x_1, \dots, x_k\} \to \{0, 1\}$ putting $f_T(x_i) := 1$ if $i \in T$, and 0 otherwise. Let $\bar{f}_T$ be the homomorphism onto $\mathbf{2}$ extending f_T. For any $T \in \mathcal{P}(k)$ and any $\mu \in \mathbf{Cm}\,\mathbf{F}_n(k)$ we have $\mu \in \mathrm{I}_{\mathbf{F}_n(k)} \iff \mu = \ker \bar{f}_T$. Next, define

$$x_T := \bigwedge_{i \in T} x_i \wedge \bigwedge_{i \notin T} x_i^*. \tag{at}$$

Then, for any $T \in \mathcal{P}(k)$, the element x_T is an atom and every atom of $\mathbf{F}_n(k)$ is of this form. Therefore, if $\mu \in \mathrm{I}_{\mathbf{F}_n(k)}$ then $1/\mu = [x_T)$ for some $T \in \mathcal{P}(k)$. Write μ_T for that μ.

It can be shown that each join-irreducible element $p \in \mathbf{F}_n(k)$ is the smallest element of $1/\mu$ for some $\mu \in \mathbf{Cm}\,\mathbf{F}_n(k)$. For an arbitrary but fixed $\mu \in \mathbf{Cm}\,\mathbf{F}_n(k)$, we define

$$L := \{i < k : x_i \in 1/\mu\}, \qquad \mathcal{T} := \{T \in \mathcal{P}(k) : \mu \subseteq \mu_T\}, \qquad p^L_{\mathcal{T}} := \big(\bigvee_{T \in \mathcal{T}} x_T\big)^{**} \wedge \bigwedge_{i \in L} x_i.$$

Intuitively, L encodes the set of generators that μ maps to 1, and $\mathcal{T}$ encodes the set of maximal congruences extending μ. For any $L \subseteq k$ and nonempty $\mathcal{T} \subseteq \mathcal{P}(k)$, such that $L \subseteq \bigcap \mathcal{T}$, we will write $\mu^L_{\mathcal{T}}$ for the unique congruence in $\mathbf{Cm}\,\mathbf{F}_n(k)$ such that $1/\mu^L_{\mathcal{T}} = [p^L_{\mathcal{T}})$.

Definition 2.2. *Let $\mathcal{T}$ and $\mathcal{S}$ be nonempty subsets of $\mathcal{P}(k)$. Let $L \subseteq \bigcap \mathcal{T}$ and $K \subseteq \bigcap \mathcal{S}$. Define an ordering relation $\leq^{\mathbf{Cm}}$ on $\mathbf{Cm}\,\mathbf{F}_n(k)$ putting*

$$\mu^L_{\mathcal{T}} \leq^{\mathbf{Cm}} \mu^K_{\mathcal{S}} \iff \mathcal{S} \subseteq \mathcal{T} \text{ and } L \subseteq K.$$

Theorem 2.3 (Structure of free p-algebra)**.** *We have:*

$$\mathbf{F}_n(k) \cong \mathrm{Up}(\mathbf{Cm}\,\mathbf{F}_n(k), \leq^{\mathbf{Cm}})$$

where Up *is the usual up-set operator.*

Theorem 2.4 (Normal form theorem)**.** *Every element t of the algebra $\mathbf{F}_n(k)$ is of the form*

$$t = \bigvee \max\{p^L_{\mathcal{T}} \in \mathcal{J}(\mathbf{F}_n(k)) : p^L_{\mathcal{T}} \leq t\}$$

where $\mathcal{J}(-)$ stands for the set of join-irreducible elements.

For $n \geq \mathcal{P}(k)$, this yields $|\mathcal{J}(\mathbf{F}_n(k))| = \sum_{i=0}^{k} \binom{k}{i}(2^{2^i} - 1)$, a formula known before, but our calculation is much easier.

3 Subquasivarieties

Using our description of free algebras and a few tricks we can show that

- Each free p-algebra belongs to the splitting companion (in the quasivariety lattice) of Pa_3. For $n \geq 2$, the interval $[\mathsf{Pa}_n, \mathsf{Pa}_{n+1}]$ is of cardinality continuum.
- For $n \geq 3$, the variety Pa_n is not structurally complete in the algebraic sense, in spite of the fact that the corresponding logic is structurally complete by a result of G. Mints. The discrepancy is due to non-algebraizability of the logic.

Hereditarily Structurally Complete Extensions of **R**-mingle

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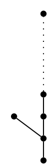
Abstract

The presentation is devoted to structurally complete extensions of the system **R**-mingle. The main theorem states that the set of all hereditarily structurally complete extensions of **RM** is countably infinite and 'almost' forms a chain having only one 'branching' element. As a corollary, we establish that the set of structurally complete **RM**'s extensions which are not hereditary is also countably infinite and forms a chain. We use algebraic methods to provide a full description of both sets. Additionally, we provide a certain characterization of the passive structural completeness among extensions of **RM**. Namely, we prove that a given quasivariety of Sugihara algebras is passively structurally complete iff it does not contain any of the two special algebras. As a corollary, an extra characterization of quasivarieties of Sugihara algebras which are oveflow complete but not structurally complete is given.

Extended Abstract

The presentation will be devoted to structural completeness [11] among consequence relations extending the system **R**-mingle [1]. Results on structural completeness of **RM** has been restricted either to some fragments of $\mathbf{RM}^{\mathbf{t}}$ [9, 10], or just to its axiomatic extensions [8]. We will consider **RM** in its original signature and with respect to its arbitrary (finitary and structural) extensions. Our main theorem states that the set of all hereditarily structurally complete extensions of **RM** is countably infinite and 'almost' forms a chain having only one 'branching' element. Precisely, we will prove that the structure of the poset of all hereditarily structurally complete subquasivarieties of Sugihara algebras is an ω^+ well-ordering with an additional element adjoined above number one:



RM is known to be algebraizable [3] with the quasivariety of Sugihara algebras [5]. Sugihara algebras are locally finite [2] and locally finite quasivarieties are known to be generated by theirs critical members [6]. Thus, our main tool will be critical Sugihara algebras which have been described in in [4]. To prove the main theorem, we will also use the caracterization of the bottom of the lattice of Sughihara subquasivarieties obtained in [7]. On the basis of the main result, we shall establish several corollaries. First, we will show that the set of structurally complete **RM**'s extensions which are not hereditarily structurally complete is also countably infinite and forms a chain. Additionally, we provide a certain characterization of the passive structural completeness [12] among extensions of **RM**. Namely, we prove that a given quasivariety of Sugihara algebras is passively structurally complete iff it does not contain any of the two special algebras. Also, an extra characterization of quasivarieties of Sugihara algebras which are passively structurally complete but not structurally complete is given.

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Evolution systems: amalgamation, absorption, and termination

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We introduce the concept of an abstract evolution system which provides a convenient framework for studying generic mathematical structures. The talk is based on a joint work with W. Kubiś [1].

Definition. An ***evolution system*** is a structure of the form $\mathcal{E} = \langle \mathfrak{V}, \mathcal{T}, \Theta \rangle$, where $\mathfrak{V}$ is a category, Θ is a fixed $\mathfrak{V}$-object (called the ***origin***), and $\mathcal{T}$ is a class of $\mathfrak{V}$-arrows (its elements are called ***transitions***) satisfying

$$\forall_{X \in \mathrm{Obj}(\mathfrak{V})} \quad \mathrm{id}_X \in \mathcal{T} \quad \text{and} \quad \forall_{t \in \mathcal{T}} \forall_{h \in \mathrm{Iso}(\mathfrak{V})} \quad h \circ t \in \mathcal{T}.$$

Main focus of this concept resolves around ***evolutions***, namely sequences of the form

$$\Theta \to A_0 \to A_1 \to \cdots \to A_n \to \cdots$$

where each of the arrows above is a transition. With every evolution $\vec{a}$ we will associate its colimit $\lim \vec{a} = A_\infty$. A finite composition of transitions will be called a ***path*** An object X is ***finite*** if there exists a path from the origin to X. A ***σ-path*** from A to B is the colimiting arrow in a sequence of transitions of the form $A = A_0 \to A_1 \to \cdots$ with colimit B.

Example. Let $\mathfrak{V}$ be the category whose objects are first-order structures of a fixed language, e.g., graphs, ordered sets, (semi-)groups, etc. We turn it into a natural evolution system $\mathcal{E}$. Namely, a transition from X to Y will be an embedding $t \colon X \to Y$ such that either $Y = t[X]$ or Y is generated by $t[X] \cup \{v\}$ for some $v \in Y \setminus t[X]$. Finally, we need to set the origin Θ, which typically is the trivial structure or any of the simplest finite structures of our choice.

Later on, we consider the following properties of both the whole system and of particular evolution of interest. Given a $\mathfrak{V}$-object X, we denote by $\mathcal{T}(X) = \{f \in \mathcal{T} \colon \mathrm{dom}(f) = X\}$, the class of all transitions with domain X. Arrows f, g are said to be ***left-isomorphic*** if $g = h \circ f$ for some isomorphism h. We say that $\mathcal{E}$ is ***locally countable*** if $\mathcal{T}(X)$ has countably many left-isomorphism classes for every finite object X.

We say that an evolution system $\mathcal{E}$ has ***transition amalgamation property*** (TAP) if for every pair of transitions $t_1 \colon A \to B$ and $t_1' \colon A \to C$, where A is a finite object, there exist transitions $t_2 \colon B \to D$ and $t_2' \colon C \to D$ such that $t_2 \circ t_1 = t_2' \circ t_1'$.

Similarly $\mathcal{E}$ has the ***amalgamation property*** (AP) if for every two paths f, g with $\mathrm{dom}(f) = \mathrm{dom}(g)$ finite, there exist paths f', g' such that $f' \circ f = g' \circ g$. Clearly TAP $\implies$ AP, while the converse is false.

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Let $\vec{u}$ be an evolution. We say that $\vec{u}$ has the ***absorption property (for transitions)*** if for every $n \in \omega$, for every path (transition) $t\colon A_n \to Y$ there are $m \geq n$ and a path $g\colon Y \to A_m$ such that $g \circ t = f_n^m$.

$$\begin{tikzcd}
A \arrow[r, "t_1"] \arrow[d, "t_1'"'] \arrow[dr, phantom, "\circ"] & B \arrow[d, "t_2"] \\
C \arrow[r, "t_2'"'] & D
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\vec{u}\colon U_0 \arrow[r] & \cdots \arrow[r] & U_n \arrow[r] \arrow[rr, bend left, "f_n^m"] \arrow[d, "t"'] & \cdots \arrow[r] \arrow[d, phantom, "\circ"] & U_m \arrow[r] & \cdots \\
 & & Y \arrow[urr, bend right, "g"'] & & &
\end{tikzcd}$$

TAP **Absorption**

Theorem. Assume $\mathcal{E}$ is a locally countable evolution system with the TAP. Then there exists an evolution $\vec{u}$ with the absorption property. Moreover, let U be the colimit of $\vec{u}$. Then

(1) Every finite object admits a σ-path into U.

(2) For every finite object A, for every two σ-paths f_0, f_1 from A to U there exists an automorphism $h\colon U \to U$ such that $f_1 = h \circ f_0$.

Example. Consider a category of graphs; as the origin let us take a single vertex, and a transition is adding one vertex connected with some of the already existing ones. We obtain different evolution systems depending on how this new edge is connected: to all vertices, to none of them, at random, to 10% of existing vertices, to (at most) k of them and so on. In the talk we will discuss how it influences the colimit of an evolution with the absorption property.

We end with a brief discussion on ***terminating*** evolution systems, namely systems in which every evolution is eventually trivial, that is, from some point on all transitions are isomorphisms. A finite object N is ***normalized*** if every transition from N is an isomorphism. An evolution system $\mathcal{E}$ is ***regular*** if $t \circ h \in \mathcal{T}$ is a transition whenever $t \in \mathcal{T}$ and h is an isomorphism. An evolution system $\mathcal{E}$ is ***locally confluent*** if for every two transitions f, g with $\mathrm{dom}(f) = \mathrm{dom}(g)$ finite, there exist paths f', g' satisfying $f' \circ f = g' \circ g$.

Theorem. Every regular locally confluent terminating evolution system has the amalgamation property.

Theorem. Let $\mathcal{E}$ be a regular locally confluent terminating evolution system. Then there exists a unique, up to isomorphism, normalized object U. Furthermore

(1) Every finite object admits a path into U.

(2) For every finite object A, for every two paths f_0, f_1 from A to U there exists an automorphism $h\colon U \to U$ such that $f_1 = h \circ f_0$.

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On semidirect products of biresiduation algebras

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Biresiduation algebras or pseudo-BCK-algebras are the $\{/, \backslash, 1\}$-subreducts of integral residuated po-monoids (or lattices). We will discuss semidirect products of biresiduation algebras, with emphasis on divisible biresiduation algebras.

To begin with, we recall that a *biresiduation algebra* [8] or a *pseudo-BCK-algebra* [3] is an algebra $(A, /, \backslash, 1)$ satisfying the equations $(y/x)\backslash((z/y)\backslash(z/x)) = 1$, $((x\backslash z)/(y\backslash z))/(x\backslash y) = 1$, $1\backslash x = x$, $x/1 = x$, $x\backslash 1 = 1$, $1/x = 1$, and the quasi-equation $(x\backslash y = 1 \;\&\; y\backslash x = 1) \Rightarrow x = y$. Following [9], we call a biresiduation algebra *divisible* if it satisfies the equations $(x\backslash y)\backslash(x\backslash z) = (y\backslash x)\backslash(y\backslash z)$ and $(z/x)/(y/x) = (z/y)/(x/y)$ (which in case of integral residuated po-monoids are equivalent to the divisibility law). By a *closure endomorphism* we mean an endomorphism that is also a closure operator.

Given two nontrivial biresiduation algebras C, D and an action ρ of C on D we define the *semidirect product* $C \ltimes_\rho D$ to be $\{(a, x) \in C \times D \colon \rho(a, x) = x\}$ with $(a, x)\backslash(b, y) = (a\backslash b, x\backslash\rho(a, y))$ and $(b, y)/(a, x) = (b/a, \rho(a, y)/x)$. If the action ρ satisfies certain conditions resembling divisibility and the maps $\rho(a, -)$ are closure endomorphisms of D, then $C \ltimes_\rho D$ is a biresiduation algebra (with a closure endomorphism), and $C \ltimes_\rho D$ is divisible if and only if both C and D are divisible.

This construction is a quite straightforward generalization of *symmetric semidirect products* of the so-called CKL-algebras [6] (which are equivalent divisible BCK-algebras or HBCK-algebras [1]) as well as of *quasidirect products* of Hilbert algebras [2]. In fact, similarly to [4], it goes back to the construction of implicative semilattices from triples consisting of a boolean algebra, an implicative semilattice and an *admissible function* [5].

If A is a divisible biresiduation algebra with a fixed closure endomorphism δ, then $C = \delta(A)$ is a subalgebra of A, $D = \delta^{-1}(1)$ is a filter of A (hence a biresiduation algebra) and, for every $a \in C$, the map $\rho(a, -) = a\backslash -$ is a closure endomorphism of D. Thus we can construct the semidirect product $C \ltimes_\rho D$. Though A is in general smaller than $C \ltimes_\rho D$, the two algebras determine essentially the same triples. In some particular cases, $C \ltimes_\rho D$ is isomorphic to A. For example, this happens when A is a BL-algebra and the fixed closure endomorphism δ is just the double negation (this generalizes the results of [4]).

For divisible biresiduation algebras, we have an adjunction between the category of algebras with closure endomorphisms and the category of "modules"/triples. Specifically, (i) let $\mathcal{A}$ be the category of divisible biresiduation algebras with fixed closure endomorphisms, i.e., algebras (A, δ), with morphisms = homomorphisms, and (ii) let $\mathcal{M}$ be the category of "modules" D over C, i.e., triples (C, D, ρ) where C, D are divisible biresiduation algebras and ρ an action of C on D, with morphisms from (C, D, ρ) to (C_1, D_1, ρ_1) defined as pairs of homomorphisms $f \colon C \to C_1$, $g \colon D \to D_1$ such that $g(\rho(a, x)) = \rho_1(f(a), g(x))$ for all $a \in C$ and $x \in D$. Then, using the assignments "algebra $(A, \delta) \mapsto$ triple (C, D, ρ)" and "triple $(C, D, \rho) \mapsto$ semidirect product $C \ltimes_\rho D$" described above, we define adjoint functors $F \colon \mathcal{A} \to \mathcal{M}$ and $G \colon \mathcal{M} \to \mathcal{A}$, with $F \dashv G$.

We will also discuss the role of n-potent elements and characterize the so-called *quasi-decompositions* (in the sense of [7] or [2]) corresponding to closure endomorphisms of divisible biresiduation algebras.

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Canonical Extensions of Quantale Enriched Categories

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Introduction. We extend canonical extensions from the ordered to the quantale enriched setting. We pay particular attention to *non-commutative* quantales and develop our work in a language that stays as faithful as possible to both order theory [4] and category theory [10]. The reason is not only to make our work accessible to both communities: In our own ongoing work, we need to have easy access to general category theoretic results and to an algebraic language in the style of lattice theory and logic.

Related Work. [2] defines the MacNeille completion of a relation $\mathrm{I} : X \times A \to 2$. Our interest stems from generalising Formal Context Analysis [6] with its applications to data bases and data analysis to the fuzzy and many-valued setting [9, 3]. The MacNeille completion of quantale enriched categories has been studied in [11, 7, 5].

Quantales. A quantale $(\Omega, \sqsubseteq, \bigsqcup, e, \cdot,)$ is a complete join semilattice $(\Omega, \sqsubseteq, \bigsqcup)$ and a monoid $(\Omega, e, \cdot)$ in which multiplication distributes over joins. We write top as $\top$ and bottom as $\bot$. Since Ω is complete it also has meets $\bigsqcap$. Multiplication has a left-residual $\triangleleft$ and the right-residual $\triangleright$ defined as $b \sqsubseteq a \triangleright c \Leftrightarrow a \cdot b \sqsubseteq c \Leftrightarrow a \sqsubseteq c \triangleleft b$.

Examples. (a) The *two-chain* $2 = \{0 \sqsubseteq 1\}$ is a commutative quantale.
(b) The *Lawvere quantale* $[0, \infty]$ is a subset of the extended real numbers. It is ordered by $\geq$ with top $\top = 0$ and has $+$ as multiplication. The residual is truncated minus $a \triangleleft b = a \dot{-} b$.
(c) The *quantale of languages* $\mathcal{P}(\Sigma^*)$ is given wrt a set Σ and has as elements subsets of the set of finite words Σ^*. Multiplication is $L \cdot L' = \{vw \mid v \in L, w \in L'\}$ where vw denotes the concatenation of the words. The residuals are given by $L \triangleright M = \{w \in \Sigma^* \mid \forall v \in L \,.\, vw \in M\}$ and $M \triangleleft L = \{w \in \Sigma^* \mid \forall v \in L \,.\, wv \in M\}$.

Quantale Spaces. We call a category enriched over a quantale a *quantale space*. Enrichment over 2 gives preorders, enrichment over $[0, \infty]$ gives generalized metric spaces [8], enrichement over $\mathcal{P}(\Sigma^*)$ gives generalized non-deterministic automata (without designated initial and final states) [1].

Weighted Downsets and Upsets. A *relation* (also known as bimodule, profunctor, distributor) $R : X \looparrowright Y$ between quantale spaces X and Y is a function $X \times Y \to \Omega$ satisfying $X(x', x) \cdot R(x, y) \sqsubseteq R(x', y)$ and $R(x, y) \cdot Y(y, y') \sqsubseteq R(x, y')$. A *presheaf* (or weighted downset) $\phi \in \mathcal{D}X$ is a relation $X \looparrowright 1$. A *co-presheaf* (or weighted upset) $\psi \in \mathcal{U}Y$ is a relation $1 \looparrowright Y$. The homs are defined by $\mathcal{D}X(\phi, \phi') = \bigsqcap_{x \in X} (\phi x \triangleright \phi' x)$ and $\mathcal{U}A(\psi, \psi') = \bigsqcap_{a \in A} (\psi a \triangleleft \psi' a)$.

Canonical Extension. The canonical extension C^δ of a quantale space C is the MacNeille completion of the relation $\mathrm{I} : \mathcal{U}'C \looparrowright \mathcal{D}'C$ given by $\mathrm{I}(f, i) = \bigsqcup_c f(c) \cdot i(c)$, that is, the set of fixed points of the adjunction given by $\phi \blacktriangleright \mathrm{I} = \bigsqcap_f \phi(f) \triangleright \mathrm{I}(f, -)$ and $\mathrm{I} \blacktriangleleft \psi = \bigsqcap_i \mathrm{I}(-, i) \triangleleft \psi(i)$.

$$
\begin{tikzcd}
\mathcal{DU}'C \arrow[rr, bend left, "-\blacktriangleright \mathrm{I}"] \arrow[rr, phantom, "\perp"] \arrow[dr, dashed, bend right] & & \mathcal{UD}'C \arrow[ll, bend left, "\mathrm{I}\blacktriangleleft -"] \arrow[dl, dashed, bend left] \\
& C^{\delta} \arrow[ul, hook, "\dashv"] \arrow[ur, hook', "\dashv"'] & \\
\mathcal{U}'C \arrow[uu] \arrow[ur, "f\mapsto\overline{f}"] \arrow[rr, dotted, "\mathrm{I}"] & & \mathcal{D}'C \arrow[uu] \arrow[ul, "i\mapsto\overline{i}"'] \\
& C \arrow[ul] \arrow[ur] \arrow[uu, bend left, "{[-]}"] &
\end{tikzcd}
$$

Here $\mathcal{U}'C$ and $\mathcal{D}'C$ are subsets of $\mathcal{U}C$ and $\mathcal{D}C$ containing the representable (co)presheaves. The paradigmatic example is the set of all "weighted" filters f and "weighted" ideals i.

Theorem Let $f \in \mathcal{U}'C$ and $i \in \mathcal{D}'C$. Then C^δ is compact in the sense that[1] $C^\delta(\lim_f[-], \mathrm{colim}_i[-]) = \mathrm{I}(f, i)$. Moreover, every $(\phi, \psi) \in C^\delta$ is the colimit of a limit of C and the limit of a colimit of C.

In our talk, we will introduce an algebraic calculus for reasoning in quantale enriched categories, present the canonical extension construction, provide some examples, and discuss the extensions of functors to the canonical extensions.

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[1] $\lim_\mathrm{f}$ refers to the limit weighted by f and $\mathrm{colim}_\mathrm{i}$ to the colimit weighted by i.

On Boolean Topos Constructions by Freyd and Pataraia and their generalizations

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In 1992, A. Pitts proved a somewhat surprising result about Heyting's intuitionistic propositional calculus, IPC [3]. He showed that for each variable p and formula ϕ in IPC there exists a formula $A_p\phi$ (effectively computable from ϕ), containing only variables distinct from p which occur in ϕ, and such that for all formulas ψ not involving p, $\vdash \psi \Rightarrow A_p\phi$ if and only if $\vdash \psi \Rightarrow \phi$. Here, $\vdash$ denotes validity in IPC.

This in particular means that one can model quantification over propositional variables in IPC, which provides an interpretation of the second-order intuitionistic propositional calculus IPC^2 in IPC.

As a corollary, A. Pitts showed that a model of IPC^2 can be constructed with an algebra of truth values isomorphic to any given Heyting algebra. In [3] he also asked whether his result can be generalized further to higher order calculi.

This question can also be reformulated in topos-theoretic terms, asking whether every Heyting algebra occurs as the algebra of all subobjects of an object in a topos.

For the case when the Heyting algebra in question is in fact Boolean, the affirmative answer is the contents of Exercise 9.11 in [1]. The explicit construction of the corresponding topos $\mathcal{F}_B$ is sketched there; Johnstone attributes it to Peter Freyd.

Specifically, [1, Exercise 9.11] suggests expressing a Boolean algebra B as the (directed) union of its finite subalgebras, utilizing the fact that Boolean algebras are locally finite (finitely generated Boolean subalgebras are finite). Then one can describe the topos corresponding to B as a colimit of a directed diagram of toposes and logical functors between them, corresponding to finite subalgebras $B_0 \subseteq B$. Each B_0 is isomorphic to the powerset of the set $\mathrm{at}(B_0)$ of its atoms, and the corresponding topos is $\mathbf{Fin}^{\mathrm{at}(B_0)}$, the product of $\mathrm{at}(B_0)$ many copies of the topos $\mathbf{Fin}$ of finite sets.

We learned from the late D. Pataraia an alternative construction of what turns out to be an equivalent topos $\mathcal{L}_B$. Namely, using the Stone duality for Boolean algebras, he considered certain explicitly described subcategory of local homeomorphisms over the Stone space $X = X_B$ dual to the Boolean algebra B. Domains of his local homeomorphisms have form $(m_1 \times U_1) \sqcup \ldots \sqcup (m_n \times U_n)$, where $U_1, \ldots, U_n$ are disjoint clopen subsets of X forming a partition of X, and $m_1, \ldots, m_n$ are finite discrete spaces (can be assumed to be of pairwise distinct cardinalities). We have not heard about this kind of construction from anybody else.

To demonstrate that for a given Boolean algebra B the topos $\mathcal{F}_B$ by Freyd and the topos $\mathcal{L}_B$ by Pataraia are isomorphic, we consider a third, intermediate category $\mathcal{M}_B$. The objects of this category are pullbacks of the form shown on the right, where $E \to F$ is any map between finite discrete topological spaces, and g is any surjective continuous map from the Stone space X_B of B to F.

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"'] & E \arrow[d] \\ X_B \arrow[r, two heads, "g"'] & F \end{tikzcd}$$

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Note that later, D. Pataraia invented an entirely different construction of a topos, proving that for every Heyting algebra H, there exists a topos with the algebra of subterminals isomorphic to H [2]. However, this work was never published.

In the present work, we generalize the constructions by Freyd and Pataraia and apply the resulting generalization to some classes of Heyting algebras beyond the classes of Boolean algebras and complete Heyting algebras. Important rôle in our investigations plays the notion of *coherent object*, which we recall here.

Let A be an object of a category $\mathbf{C}$ with finite limits.

- A is *compact* if every jointly epimorphic family of subobjects of A admits a finite jointly epimorphic subfamily.
- A is *stable* if, for every pair of morphisms $U \to A \leftarrow V$ with U and V compact, the pullback $U \underset{A}{\times} V$ is compact as well.
- A is *coherent* if it is both compact and stable.

Here, a family of morphisms $(e_i \colon U_i \to A)$ is called *jointly epimorphic* if, given any two morphisms $g, h \colon A \to B$ such that $g \circ e_i = h \circ e_i$ for all i, it follows that $g = h$.

We will use coherent objects to characterize the above three categories by proving the following theorem.

Theorem. *For a given Boolean algebra B, the toposes $\mathcal{F}_B$, $\mathcal{L}_B$ and $\mathcal{M}_B$ described above are equivalent to the subcategory $\mathcal{C}$ of coherent objects in the category of sheaves $\mathbf{Sh}(X_B)$ over the Stone space X_B associated with the Boolean algebra B.*

Note in particular that coherent objects of $\mathbf{Sh}(X_B)$ form a Boolean topos.

In the talk we will discuss possible generalizations to some other classes of Heyting algebras.

One can describe the category of sheaves $\mathbf{Sh}(\mathrm{Spec}(H))$ over the spectral space $\mathrm{Spec}(H)$ corresponding to H in order-topological terms, as certain Esakia spaces over X_H. We will use this description to study analogs of the above three categories, and relate them to the subcategory of coherent objects in $\mathbf{Sh}(\mathrm{Spec}(H))$.

Finally, we consider the case of locally finite algebras and employ the latter construction of taking coherent objects in the corresponding categories of sheaves. Instead of pullbacks of maps between finite sets we will need pullbacks of local homeomorphisms between finite topological spaces. In this case the corresponding inclusion functors are no longer logical.

In the talk we will address several related questions, namely, when do coherent objects of a topos form a topos, and which spectral spaces can be obtained as inverse limits of directed diagrams of local homeomorphisms between finite spaces.

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Tensor Product in the Category of Effect Algebras and Related Categories

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Tensor product of effect algebras was studied in various articles, e.g. [JM],[JP],[Gu]. However, there are few results about which constructions and functors involving effect algebras preserve tensor products.

An important property of the category of effect algebras **EA** is that the monoidal unit and the initial object coincide. Consequently, we can consider tensoring with a fixed effect algebra E as a functor:

$$E \otimes - \colon \mathbf{EA} \to E \downarrow \mathbf{EA} \tag{1}$$

which sends an effect algebra F to a homomorphism $E \to E \otimes F$, $a \mapsto 1 \otimes a$.

Theorem 1. *For an effect algebra E, the functor $E \otimes -$ from $\mathbf{EA}$ to $E \downarrow \mathbf{EA}$ which sends F to a morphism $\iota_{E,F} \colon E \to E \otimes F$ $(a \mapsto a \otimes 1)$ admits a right adjoint $[E, -]_-$.*

Corollary 2. *Let $\mathcal{D}$ be a small connected category and $E \in \mathbf{EA}$. The functor $E \otimes - \colon \mathbf{EA} \to \mathbf{EA}$ preserves all colimits over $\mathcal{D}$.*

It turns out that several categories around **EA** share the same property. In particular, the category of ordered Abelian groups with strong unit $\mathbf{POG}_u$ and the category of partial bounded commutative monoids $\mathbf{PCM}_b$ satisfy theorems analogous to Theorem 1. Category **EA** sits between these two categories via a pair of adjunctions:

$$\begin{tikzcd} \mathbf{PCM}_b \arrow[r, bend right, "L"'] \arrow[r, phantom, "\top"] & \mathbf{EA} \arrow[l, bend right, "i"'] \arrow[r, bend right, "\mathrm{Gr}"'] \arrow[r, phantom, "\top"] & \mathbf{POG}_u \arrow[l, bend right, "\Gamma"'] \end{tikzcd} \tag{2}$$

Theorem 3. *For any $X, Y \in \mathbf{PCM}_b$ and $E, F \in \mathbf{EA}$ we have*

$$L(X \otimes Y) \cong L(X) \otimes L(Y) \text{ and } \mathrm{Gr}(E \otimes F) \cong \mathrm{Gr}(E) \otimes \mathrm{Gr}(F). \tag{3}$$

Where functors L and Gr are from (2) and the tensor products are computed in the appropriated categories.

In the case of Gr, we have even stronger result:

Theorem 4. *The left adjoint Gr in (2) extend to a strong monoidal functor.*

In the case of $\mathrm{Gr} \colon \mathbf{EA} \to \mathbf{POG}_u$, the isomorphism (3) follows from (up to isomorphism) commutativity of the diagram (4), where E is any effect algebra and $A = \mathrm{Gr}(E)$.

$$\begin{tikzcd} E \downarrow \mathbf{EA} \arrow[d, "\mathrm{Gr}"'] & \mathbf{EA} \arrow[l, "E\otimes -"'] \arrow[d, "\mathrm{Gr}"] \\ A \downarrow \mathbf{POG}_u & \mathbf{POG}_u \arrow[l, "A\otimes -"] \end{tikzcd} \tag{4}$$

The functors involved in (4) correspond to some free constructions and so are rather complicated. In the proof of commutativity, we use a trick. We move to the corresponding right adjoints (which all exist). The right adjoints all have a description in concrete terms, hence are easier to work with.

By a result in [We], the tensor product in $\mathbf{POG}_u$ does not preserve *Riesz Decomposition Property* (RDP) in general. Whereas in $\mathbf{PCM}_b$, the tensor product does preserves (RDP). The case of effect algebras was an open problem for a while. Thanks to Theorem 3, we can lift the contra-example, which works in $\mathbf{POG}_u$, to $\mathbf{EA}$.

Theorem 5. *In* $\mathbf{EA}$*, tensor product does not preserves Riesz Decomposition Property in general.*

Theorem 5 has the following implications:

- Computing tensor products in $\mathbf{EA}$ is rather hard, in the sense we cannot control it using (RDP). That is in contrast to the construction of a universal group (functor Gr), which preserves (RDP).
- The functor $L\colon \mathbf{PCM}_b \to \mathbf{EA}$, which essentially forces cancellation property, does not preserve (RDP).

It is not well understood which tensor products are preserved by the right adjoints in (2). However, it is proved in [Pu] that functor Γ preserves the tensor product of $(\mathbb{R}, 1)$ with itself, that is

$$\Gamma(\mathbb{R} \otimes \mathbb{R}, 1 \otimes 1) \cong [0,1] \otimes [0,1]. \tag{5}$$

The question of whether the embedding $i\colon \mathbf{EA} \hookrightarrow \mathbf{PCM}_b$ preserves the tensor product of the real unit interval $[0,1]$ (seen as an effect algebra) with itself leads to an interesting combinatorial problem. In the case of $\mathbf{PCM}_b$, it holds that two tensors $a_1 \otimes b_1 + \cdots + a_n \otimes b_n$ and $c_1 \otimes d_1 + \cdots + c_m \otimes d_m$ in $[0,1] \otimes [0,1]$ are equal if and only if we can represent the two tensors as two orthogonal polygons $\mathcal{P}_1$ and $\mathcal{P}_2$ inside the unit square $[0,1] \times [0,1]$, and there is an orthogonal dissection between $\mathcal{P}_1$ and $\mathcal{P}_2$. By a result in [Ep], there is a full Dehn invariant for this kind of dissection. We have used this result to show that computing the tensor product of the real unit interval with itself as a partial monoid in $\mathbf{POG}_b$ and as an effect algebra in $\mathbf{EA}$ is essentially equivalent.

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Subdirectly irreducible and generic equational states

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An Abelian lattice-ordered group (ℓ-group, for short) is an Abelian group G endowed with a lattice order that is translation invariant. An ℓ-group is called unital if it contains an element u, such that for any positive $g \in G$ there exists a natural number n for which the n-fold sum of u exceeds g. A *state* of a unital ℓ-group is a normalized and positive group homomorphism in $\mathbb{R}$. It is well known that states correspond to expected-value operators on bounded real random variables. Unital ℓ-groups are not first-order definable, yet they are categorically equivalent to the equational variety of MV-algebras [1]. Thus, states can be studied in an equational setting by looking at their counterpart in MV-algebras, as first proposed in [9]. However, since states on MV-algebras are defined as particular maps into the real unit interval $[0, 1]$, a completely algebraic characterization was still missing.

Efforts to find an algebraic theory of states continued in [5] (see also [2]). There the authors introduced the notion of *internal state* as an additional unary operation with specific axioms relating it to the other MV-operations. This framework was used to provide an algebraic treatment of the Lebesgue integral. A drawback of this approach is that an internal state can be applied to itself. More recently, a different approach has been proposed. In [6] the authors first extend Mundici's equivalence between unital ℓ-groups and MV-algebras to an equivalence between states between ℓ-groups and states between MV-algebras. Secondly, they introduce the class of *equational states* as a two-sorted variety of algebras. An equational state $(\mathbf{A}_1, \mathbf{A}_2, s)$ is a two-sorted algebra in which each sort $\mathbf{A}_1$ and $\mathbf{A}_2$ is an MV-algebra with customary operations and the *state-operation* s has $\mathbf{A}_1$ as domain and $\mathbf{A}_2$ as codomain. This approach opens the way to studying probabilistic notions with algebraic tools; for instance, [6, Theorem 4.1] gives a characterization of free equational states.

Another reason for considering the class of equational states is that they provide an algebraic semantics to the probabilistic logic FP(Ł, Ł). The system FP(Ł, Ł) is a two-layer logic introduced in [4] to provide a formal framework to deal with the probability of vague events. If a vague event is codified by a formula φ in Łukasiewicz logic, its probability is given by the formula $\square(\varphi)$, which is a Łukasiewicz atomic formula interpreted as "φ is probable".

An adaptation of the classical Lindenbaum-Tarski construction produces an equational state $\mathcal{ES}_{\mathsf{Var}}$ with the following properties.

Theorem 1 ([7, Theorem 8]). *Let Var be a (one-sorted) set of propositional variables. For any FP(Ł, Ł) formula Φ, the following are equivalent.*

1. *Φ is a theorem.*
2. *Φ is valid in the equational state $\mathcal{ES}_{\mathsf{Var}}$.*
3. *Φ is valid in all equational states.*

Corollary 1 ([7, Theorem 15]). *The equational state $\mathcal{ES}_{\mathsf{Var}}$ is the free equational state generated by $(\mathsf{Var}, \emptyset)$.*

*Speaker.

We present here a continuation of the algebraic study of equational states started in [8], where it is proven that the lattice of ideals and the lattice of congruences of any equational state are isomorphic (see [8, Corollary 2]). This isomorphism enables us to characterize the subdirectly irreducible equational states as follows.

Theorem 2. *An equational state* $(\mathbf{A}_1, \mathbf{A}_2, s)$ *is subdirectly irreducible if and only if one of the following is true:*

1. $\mathbf{A}_2 = \emptyset$ *and* $\mathbf{A}_1$ *is a subdirectly irreducible MV-algebra.*
2. $\mathbf{A}_2$ *is a subdirectly irreducible MV-algebra, and the state-operation is faithful, i.e.* $s(x) = 0$ *implies* $x = 0$.

Combining the characterization of subdirectly irreducible equational states with some ideas of [3] we prove that two notable classes generate the variety of equational classes.

Theorem 3. *The following classes generate the variety of equational states:*

1. *The class of all equational states of the type* $([0,1]^W, [0,1])$*, with* W *an arbitrary set.*
2. *The class of finite equational states, i.e. equational states whose universe is finite in each sort.*

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The finite model property for lattice based $\mathsf{S4}$

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Distributive modal logics based on classical, positive and intuitionistic logics have been thoroughly investigated (see e.g. [3, 4, 10]). Non-distributive modal logics have received less attention, even though they contain important logics such as quantum logic [5, 8] and substructural logics [7]. Recently the duality and Sahlqvist theory of non-distributive modal logics were studied in [2, 9, 6]. In this abstract we investigate the finite model property of non-distributive modal logics, including a non-distributive version of $\mathsf{S4}$, from an algebraic perspective.

Let $\ell\mathsf{S4}_\Box$ be the logic whose algebraic semantics is given by lattices with a $\Box$ satisfying:

$$\Box 1 \approx 1, \qquad \Box(a \wedge b) \approx \Box a \wedge \Box b, \qquad \Box a \leq a, \qquad \Box a \leq \Box\Box a.$$

Theorem 1. *The logic $\ell\mathsf{S4}_\Box$ has the finite model property.*

Proof. Let A be an algebra with valuation $\sigma\colon Fm \to A$ such that $\sigma(\phi) \neq \sigma(\psi)$ (i.e. $A \not\models \phi \approx \psi$). We construct a finite algebra B such that $B \not\models \phi \approx \psi$. Let Σ be the set of subformulas of ϕ and ψ. Define B to be the smallest $0, 1, \Box, \wedge$-subreduct of A containing $\sigma[\Sigma]$. Then B is finite because $\Box$ is normal and $\mathsf{S4}$. Hence it is complete, so we can define a join in B as

$$a \vee_B b = \bigwedge\{c \in B \mid c \geq a, b\}.$$

One easily checks that if $a, b, a \vee b \in B$, then $a \vee_B b = a \vee b$. Therefore, we can define a valuation $\tau\colon Fm \to B$ by setting $\tau(\chi) = \sigma(\chi)$ if $\chi \in \Sigma$, and extending it to Fm in the natural way. This is well-defined. Indeed, if $\alpha \vee \beta \in \Sigma$, then

$$\tau(\alpha \vee \sigma) = \sigma(\alpha \vee \sigma) = \sigma(\alpha) \vee \sigma(\beta) = \tau(\alpha) \vee \tau(\beta) = \tau(\alpha) \vee_B \tau(\beta),$$

since $\tau(\alpha), \tau(\beta), \tau(\alpha) \vee \tau(\beta) \in B$. The valuation τ is such that $\tau(\phi) \neq \tau(\psi)$. Therefore $\phi \approx \psi$ can be refuted in a finite algebra. $\Box$

We highlight the difference with the classical cases. When proving the finite model property for classical modal logic, one would take B to be the Boolean algebra generated by Σ, and define a suitable box on it. In our case, we cannot consider the lattice generated by Σ, as it could be infinite. Instead, we generate B as a meet-semilattice. Dropping joins from the generating set allows us to add box instead (provided that it is $\mathsf{S4}$), which simplifies the construction.

Next we add a monotone diamond, in line with [2, Section 4], which satisfies the following:

$$\Diamond 0 \approx 0, \qquad \Diamond(a \vee b) \geq \Diamond a \vee \Diamond b.$$

The resulting logic is denoted by $\mathcal{L}_{\Diamond^m}\mathsf{S4}_\Box$. From this we obtain the logic $\mathcal{L}\mathsf{S4}_{\Box\Diamond^m}$ by adding:

$$a \leq \Diamond a \qquad \Diamond\Diamond a \leq \Diamond a.$$

Theorem 2. *The logics $\mathcal{L}_{\Diamond^m}\mathsf{S4}_\Box$ and $\mathcal{L}\mathsf{S4}_{\Box\Diamond^m}$ have the finite model property.*

Proof. We proceed as in the previous proof. The only difference is that we need to define a diamond on B. We define

$$\Diamond_B a = \bigwedge\{b \in B \mid b \geq \Diamond a\} \qquad \text{and} \qquad \Diamond_B a = \bigwedge\{\Diamond b \mid b \in B, \Diamond b \geq \Diamond a, \Diamond b \in B\}$$

in the first and second cases, respectively. One easily checks that if $a, \Diamond a \in B$, then $\Diamond_B a = \Diamond a$. One can also check that $\Diamond_B$ is monotone, and that it is $\mathsf{S4}$ provided that $\Diamond$ is. $\Box$

One might wonder if the diamond can be made normal. The main difficulty lies in non distributivity. In [1, Lemmas 4.5 & 6.2], the proofs rely on distributivity. Another difficulty arises from the fact that one does not need to prove $\Diamond_B(a \vee b) \leq \Diamond_B a \vee \Diamond_B b$, but $\Diamond_B(a \vee_B b) \leq \Diamond_B a \vee_B \Diamond_B b$.

So far, we have treated $\Box$ and $\Diamond$ as two unrelated operators. Guided by [2, Section 4], we may wish to add interaction axioms, such as

$$\Box a \wedge \Diamond b \leq \Diamond(a \wedge b).$$

However, the method for obtaining finite models used above does not readily work in presence of this interaction axiom. We will illustrate where it fails; resolving this is ongoing work. Let A be the lattice N_∞ equipped with an identity box and a diamond sending n to $n+1$ (and sending $\top, x, \bot$ to themselves). The axiom $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ is satisfied in A. However, it cannot be satisfied in any $B \subseteq A$. Indeed, let n be the maximum of $B \cap \mathbb{N}$. Then $\Diamond n = n + 1$. In line with [1, Theorem 4.2], we wish to have $\Diamond_B m \geq \Diamond m$, which forces $\Diamond_B m = \top$. Then $\Box x \wedge \Diamond_B m = x$, although $\Diamond_B(x \wedge m) = \Diamond_B \bot = \bot$. Therefore, $\Box a \wedge \Diamond_B b \leq \Diamond_B(a \wedge b)$ is refuted in B.

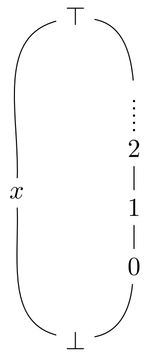

Figure 1: The lattice N_∞

This leaves the finite model property of this logic as an open question. We intend to resolve it by exploring the Kripke-like semantics of non-distributive modal logic developed in [2].

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On Three-valued Coalgebraic Cover Modalities*

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Coalgebraic logic proposed by Moss uses a single modality ∇_T with a set endo-functor T as its arity and T-coalgebras as structural frames [11]. Finitary version of Moss' coalgebraic logic has found applications in logic and automata theory [9], its soundness and completeness has been established in a form of Hilbert-style axiomatization [8], and the Gentzen-syle sequent calculi [4]. An adaptation of Moss' coalgebraic logic to a many-valued context, where formulas are evaluated in a given algebra of truth values, has been explored *e.g.* in [2] and [3]. Nevertheless, proof theory for many-valued coalgebraic cover modality has not been discussed yet. Our aim is to bridge this gap by proposing a Gentzen-style sequent calculus for a three-valued coalgebraic cover modality, expanding Kleene logic. Besides, we will also touch upon utilizing the abstract approach as in [10] and propose possible axioms for Hilbert-style systems over semi-primal algebras such as 3-valued Łukasiewicz chain.

We start with choosing, as the propositional base, Strong Kleene logic (K_3), Weak Kleene logic (WK_3), which arise from different algebras (matrices) on the three values $\{1, n, 0\}$, with varying interpretation of the third value n (undefined, nonsensical, paradoxical) [7, 6]. Consequence relations of these logics can be closely related to classical consequence. In case of WK_3 where the three-element algebra is not a lattice and the third value n is infectious it is done using certain variable containment conditions. This allows for a natural adaptation of classical sequent calculus where some rules use variable containment side conditions [5]. We show how these conditions can be modalized and use it to built on sequent calculi for coalgebraic cover modality.

As a starting example, consider $\mathcal{P}$ to be the (covariant) power set functor and $\mathcal{P}_\omega$ be the finitary power set functor. Let $\mathcal{L}_{K_3}$ be the following language:

$$\varphi \coloneqq p \mid \bigvee \Phi \mid \bigwedge \Phi \mid \neg\varphi \mid \nabla\alpha \mid \Delta\alpha$$

where $p \in Prop$, a set of propositional variables, and $\Phi, \alpha \in P_\omega \mathcal{L}_{K_3}$. The set $Var_i(\Phi)$ denotes propositional variables within formulas of modal depth i in Φ, and $Base^{\mathcal{P}_\omega}_{\mathcal{L}_{K_3}}(\alpha)$ is defined as $\bigcap\{X \subseteq_\omega \mathcal{L}_{K_3} \mid \alpha \in \mathcal{P}_\omega X\}$. The semantics for the logical connectives of $\mathcal{L}_{WK_3}$ can be defined using the truth tables in Weak Kleene logic. The semantics for $\nabla\alpha$ is defined as follows:

Definition 1. *Let S be a set. For a coalgebra $\sigma : S \to \mathcal{P}(S)$ together with the atomic evaluation $ev : S \times Prop \to \{0, 1, n\}$*

$$s \Vdash^w_\sigma \nabla\alpha \coloneqq \sigma(s)\hat{\mathcal{P}}(\Vdash^w_\sigma)(\alpha) = \bigwedge_{t\in\sigma(s)} \bigvee_{a\in\alpha} t \Vdash^w_\sigma a \wedge \bigwedge_{a\in\alpha} \bigvee_{t\in\sigma(s)} t \Vdash^w_\sigma a$$

where $\hat{\mathcal{P}}(\Vdash^w_\sigma)$ is the power set relation lifting of $\Vdash^w_\sigma$.

The infectious property of Weak Kleene logic implies that if there exist some $t \in \sigma(s)$ and $a \in \alpha$ such that $t \Vdash^w_\sigma a = n$ then $s \Vdash^w_\sigma \nabla\alpha = n$. Otherwise the $\Vdash^w_\sigma$ relation acts the same as in the classical case. The Genzen sequent calculi GWK_3 employs the following modal depth-specific side conditions as in [5]. For example, the ($\neg$-r) rule would now become:

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$$\frac{\Gamma, a \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma, \neg a}\ (\neg\text{-r}),\ \forall i \leq m,\ Var_i(a) \subseteq Var_i(\Gamma)$$

$$\frac{\{A_L^\Phi \Rightarrow A_R^\Phi \mid \Phi \in SRD(\Gamma \uplus \Sigma)\}}{\{\nabla\alpha \mid \alpha \in \Gamma\} \Rightarrow \{\Delta\beta \mid \beta \in \Sigma\}}\ \forall\Phi.A^\Phi \in Base(\Phi), \forall i \leq n,\ Var_i(A_L^\Phi) \subseteq Var_i(A_R^\Phi)$$

where m is the maximum modal depth of formulas in Γ, and n is a maximum modal depth of formulas in $\Gamma \cup \Sigma$. In this talk, we will discuss how to obtain a Gentzen system for the coalgebraic cover modality over Weak Kleene logic.

For the Strong Kleene logic, the semantics for logical connectives in $\mathcal{L}_{K_3}$ is defined via the truth tables in Strong Kleene logic, and the modal formula $\nabla\alpha$ is defined similarly to Definition 1. The Genzen sequent calculus GK_3 is based on GWK_3, obtained by removing all the side conditions and adding six negation related rules [1]. We will show how to extend the calculus with the ∇-modality rules. We will then discuss soundness and completeness of the resulting calculi. As [10] indicates, completeness can be lifted from the classical logic to the many-valued logic in case the algebras are semi-primal. Nevertheless, since the semantic for Weak Kleene logic is not semi-primal, the approach in [10] is not feasible here.

In the end of this talk, we will briefly address the problem of axiomatizing semi-primal algebra-valued coalgebraic logic by demonstrating when modifying the modal axioms (∇1)-(∇4) in [8] and adding the following axioms results in a sound Hilbert-style axiomatic system:

$$\tau_v(\nabla\Phi) \equiv \nabla T(\tau_v)(\Phi),$$

where v are elements of semi-primal algebras $\mathbb{A}$ and for τ_v are unary operations defined by

$$\tau_v(x) = \begin{cases} 1, & \text{if } x \geq v \\ 0, & \text{if } x \ngeq v. \end{cases}$$

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Equivalential Algebras with Regular Semilattice

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Abstract

Consider two operations on a Heyting algebra: $x\widehat{\wedge}y = \neg\neg x \wedge \neg\neg y, x \cdot y = x \to y \wedge y \to x$ and a class $\mathcal{H}$ of all $(\cdot, \widehat{\wedge})$ subreducts of Heyting algebras. It is easy to see that $\mathcal{H}$ is a quasivariety and all algebras in this class are Fregean (1-regular and congruence orderable) and congruence permutable. We show that $\mathcal{H}$ is actually a variety by characterizing it as an equational class. In the language of Tame Congruence Theory this class is an example of a mixed type; algebras can behave locally both like a finite vector space over a finite field and like a two element boolean algebra.

1 Introduction

According to [4], there are only finitely many polynomialy nonequivalent algebras of given fixed size which generate a congruence permutable (CP) Fregean variety. Because (in CP Fregean) the clone of polynomials is determined by the congruence lattice suplemented by the commutator operator, it is easy to check that on a three element universe there are exactly four such nonequivalent algebras. Each of them can be obtained by taking an appropriate reduct of a three-element Heyting algebra. Two of those are from well known classes: equivalential algebras and Brouwerian semilattices, however the other two are not. We denote those $\mathbf{R}$ and $\mathbf{D}$. An interesting property of $\mathbf{R}, \mathbf{D}$ is that unlike equivalential algebras or Brouwerian Semilattices they both have a mixed type in the language of Tame Congruence Theory [1]. This lead to research of varieties generated by $\mathbf{R}$ and $\mathbf{D}$ done by Sławomir Przybyło in his PhD thesis (published in [5, 6]).

Because $\mathbf{R}$ can be obtained as a $(\cdot, \widehat{\wedge})$ reduct of a Heyting algebra we went on to investigate the class $\mathcal{H}$ of all $(\cdot, \widehat{\wedge})$-subreducts (of which $\mathcal{V}(\mathbf{R})$ is a proper subclass). Similar research was already done for classes of $(\cdot, \neg)$ and $(\cdot, \neg\neg)$ subreducts leading to a characterization of the first one as a quasivariety and the other as a variety [3, 2]. We follow a similar path by first "guessing" an equational class and then incrementaly showing its properties until we arrive at a conclusion that it is in fact the class of all subreducts.

2 EARS

Definition 1. An algebra $\mathbf{A} = (A, \cdot, \widehat{\wedge})$ with two binary operations is called an *equivalential algebra with regular semilattice* or *EARS* if a series of identities are satisfied for any $x, y, z \in A$. Before we write them down, for the sake of brevity, we extend our language by adding an unary operation $r(x) = x\widehat{\wedge}x$ and adopt a convention that $\cdot$ is associating to the left $(xyz = (xy)z)$. The identities are as follows:

E1. $xxy \approx y$;

E2. $xyzz \approx (xz)(yz)$;

E3. $xy(xzz)(xzz) \approx xy$;

S1. $r(x)\widehat{\wedge}y \approx x\widehat{\wedge}y$;

S2. $x\widehat{\wedge}y \approx y\widehat{\wedge}x$;

S3. $(x\widehat{\wedge}y)\widehat{\wedge}z \approx x\widehat{\wedge}(y\widehat{\wedge}z)$;

M1. $r(x)yy \approx r(x)$;

M2. $r(xx) \approx xx$;

M3. $x(y\widehat{\wedge}z)(y\widehat{\wedge}z) \approx xr(z)r(z)r(y)r(y)$;

M4. $(x\widehat{\wedge}z)(y\widehat{\wedge}z)r(z) \approx (xy)\widehat{\wedge}z$;

M5. $xr(x)r(x) \approx x$.

Identities E1-E3 make $(A, \cdot)$ an equivalential algebra and identities S1-S3 impose a semilattice structure on $(r(A), \widehat{\wedge})$. The remaining five identities describe how those two objects are mixed together.

Of course $\mathcal{H}$ is a subclass of the variety of all EARS $\mathcal{V}_{EARS}$. We will start with some basic properties of the operations and congruences of EARS and then show the following facts:

Lemma 1. $\mathcal{V}_{EARS}$ is congruence orderable.

As 1-regularity and congruence permutability is preserved by reducts it follows that

Lemma 2. $\mathcal{V}_{EARS}$ is congruence permutable Fregean.

Lemma 3. $\mathcal{V}_{EARS}$ is locally finite.

And the main result

Theorem 1. *Every finite EARS is in $\mathcal{H}$, which leads to $\mathcal{V}_{EARS} = \mathcal{H}$.*

We will also present some results about the commutator in EARS and the structure of subvarieties of $\mathcal{V}_{EARS}$.

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Localic uniform completions via Cauchy sequences

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Uniform spaces provide a general setting in which to discuss uniform continuity and completeness. A uniform space is given by a set X equipped with a filter $\mathcal{E}$ of binary relations on X, called *entourages*, satisfying certain axioms. (For instance, see [4, Chapter 9].) Entourages intuitively act like approximate equality relations. Every metric space (X, d) gives rise to a uniform space with basic entourages $E_\varepsilon = \{(x, y) \in X \times X \mid d(x, y) < \varepsilon\}$.

Recall that the completion of a metric space can be constructed as a quotient of the set of Cauchy sequences. However, this does not work for general uniform spaces. The problem is that sequences are not always 'long enough' and so we need to use *Cauchy filters* or *Cauchy nets* instead.

The pointfree approach to uniformity replaces uniform spaces with *uniform locales*. Analogously to before, a uniform locale is a locale X equipped with a filter of open sublocales of X^2 satisfying certain conditions and there is a well-developed theory of completions of uniform locales via Cauchy filters. See [2] or [1] for details.

In this talk, we will show that, in contrast to the situation with uniform spaces, the correct completion of uniform locales *can* also be obtained using Cauchy sequences. Our construction is based on the construction of the so-called 'localic completion' of metric spaces in terms of Cauchy sequences described by Vickers in [3], but generalises it to start with locales rather than sets and to use uniform rather than metric structures.

We must first construct a locale of Cauchy sequences. Usually, we would obtain this as the classifying locale of a *geometric theory* of Cauchy sequences. Recall that a Cauchy sequence in a uniform space $(X, \mathcal{E})$ is a map $s\colon \mathbb{N} \to X$ such that $\forall E \in \mathcal{E}.\ \exists N \in \mathbb{N}.\ \forall n, n' \geq N.\ (s(n), s(n')) \in E$. The problem is that this definition involves universal quantification over $\mathbb{N}$ and so is too logically complex to be described by a geometric theory. Vickers circumvents this by asking the Cauchy sequences to converge rapidly, but rapid convergence cannot be defined outside of the metric setting.

Instead we 'Skolemise' the definition to reverse the quantifiers and give $\exists m\colon \mathcal{E} \nrightarrow \mathbb{N}.\ \forall E \in \mathcal{E}.\ \forall n, n' \geq m(E).\ (s(n), s(n')) \in E$ and then include the *modulus of Cauchyness* m in the data of a Cauchy sequence. (This additional data will be discarded by the quotient step in any case.) This allows us to define a locale of *modulated Cauchy sequences*.

Finally, we construct a map from this locale to the usual completion and prove it is a well-behaved quotient map. Thus, the completion can indeed be obtained as a quotient of the locale of (modulated) Cauchy sequences.

A natural question is now: what goes wrong in the spatial setting? The problem is that, unless the uniformity has a countable base, the locale of modulated Cauchy sequences is unlikely to be spatial. The spatial construction can be understood as taking the points of this locale *before* taking the quotient. To obtain the correct completion of a uniform space we must instead take points *after* taking the quotient. Thus, the root of the pathologies that occur in the spatial setting is that taking the spectrum of a locale does not preserve quotients!

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Categorical Continuous Logic

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Continuous logic is obtained by replacing the binary truth values $\{0, 1\}$ by the unit interval $[0, 1]$. It was introduced for the model theory of complete metric structures, see [5] for a recent introduction. I will explain how continuous logic arises naturally when combining categorical logic and duality theory.

A coherent hyperdoctrine is a functor $\mathbf{C}^{\mathrm{op}} \to \mathbf{DL}$ satisfying some axioms (see, e.g., [3, Ch. 5]), where $\mathbf{C}$ is a left exact category of contexts and where $\mathbf{DL}$ is the category of distributive lattices. These hyperdoctrines algebraize theories in coherent logic, more precisely the ones extending the theory of flat functors on $\mathbf{C}$. Composing with Priestley duality, one obtains a functor $\mathbf{C} \to \mathbf{Priestley}$, giving, in model theoretic terms, the spaces of types of the theory. The functors obtained in this way can be axiomatized as the *open polyadic Priestley spaces* [9]. We can replace the Priestley spaces by the more general compact ordered spaces to obtain *open polyadic compact ordered spaces* and it is possible to develop an elementary model theory from this order-topological perspective (Beth definability, omitting types, Makkai conceptual completeness).

In order to come back to the algebraic side, two dualities for compact ordered spaces behave well:

1. The duality between compact ordered spaces and stably continuous frames.
2. The duality obtained in [1, 2] by taking the unit interval $[0, 1]$ as a dualizing object.

Applying either of these dualities yields a different kind of hyperdoctrine. We will call them respectively stably continuous hyperdoctrines and fuzzy hyperdoctrines. Each possibility has its own advantage.

The duality with stably continuous frames allows to draw a connection to topos theory. The classifying toposes of stably continuous hyperdoctrines are the *stably continuous toposes*, specializing the continuous toposes of [6].

On the other hand, the duality of [1, 2] allows for a very straightforward generalization of intuitionistic logic. For instance, Pitts' uniform interpolation theorem [7] still holds by generalizing the proof of [4, 8].

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Bi-Intermediate Logics of Co-Trees: Local Finiteness and Decidability

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Bi-intuitionistic logic bi-IPC is the conservative extension of (propositional) intuitionistic logic IPC obtained by adding a new binary connective $\leftarrow$ to the language, called the *co-implication*, which behaves dually to $\to$. In this way, bi-IPC reaches a symmetry, which IPC lacks, between the connectives $\wedge, \top, \to$ and $\vee, \bot, \leftarrow$, respectively. Furthermore, thanks to the co-implication, bi-IPC achieves significantly greater expressivity than IPC. For instance, if the points of a Kripke frame $\mathfrak{M}$ are interpreted as states in time, the language of bi-IPC is expressive enough to talk about the past, something that is not possible in IPC. This feature is captured by the transparent interpretation of co-implication provided by the Kripke semantics of bi-IPC [11], since $\mathfrak{M}, x \models \phi \leftarrow \psi$ iff $\exists y \leq x$ ($\mathfrak{M}, y \models \phi$ and $\mathfrak{M}, y \not\models \psi$).

The greater symmetry of bi-IPC when compared to IPC is reflected in the fact that bi-IPC is algebraized in the sense of [3] by the variety bi-HA of *bi-Heyting algebras* [10], i.e., Heyting algebras whose order duals are also Heyting algebras. As a consequence, the lattice of *bi-intermediate logics* (i.e., consistent axiomatic[1] extensions of bi-IPC) is dually isomorphic to that of nontrivial varieties of bi-Heyting algebras. The latter, in turn, is not only amenable to the methods of universal algebra, but also from those of duality theory, since the category of bi-Heyting algebras is dually equivalent to that of *bi-Esakia spaces* [5], see also [1].

In [2], we began studying extensions of the *bi-intuitionistic Gödel-Dummett logic* bi-GD $:=$ bi-IPC $+ (p \to q) \vee (q \to p)$, the bi-intermediate logic axiomatized by the *Gödel-Dummett axiom* (also known as the *prelinearity axiom*). Over IPC, this formula axiomatizes the well-known *intuitionistic linear calculus* LC $:=$ IPC $+ (p \to q) \vee (q \to p)$ (see, e.g., [4, 6, 8, 7]). While both logics are Kripke complete with respect to the class of *co-trees* (i.e., posets with a greatest element and whose principal upsets are chains), notably, the properties of these logics diverge significantly. For example, while LC has only countably many extensions, all of which are locally finite, we proved that bi-GD is not locally finite and has continuum many extensions. Moreover, LC is also Kripke complete with respect to the class of chains, whereas we showed that the bi-intermediate logic of chains is a proper extension of bi-GD (namely, the one obtained by adding the *dual Gödel-Dummett axiom* $\neg[(q \leftarrow p) \wedge (p \leftarrow q)]$ to bi-GD). This strongly suggest that the language of bi-IPC is more appropriate to study tree-like structures than that of IPC (since we work with a symmetric language, all of our results can be dualized to the setting of trees in a straightforward manner).

One notable extension of bi-GD is $\mathit{Log}(FC) := \{\varphi \colon \forall n \in \mathbb{Z}^+ \ (\mathfrak{C}_n \models \varphi)\}$, the *logic of the finite combs* (i.e., finite co-trees whose shape resembles that of a comb, see Figure 1). We showed in [2] that if L is an extension of bi-GD, then L is locally finite iff $L \nsubseteq \mathit{Log}(FC)$. Consequently, $\mathit{Log}(FC)$ is the only pre-locally finite extension of bi-GD (i.e., it is not locally finite, but all of its proper extensions are so). More recently, we found a finite axiomatization for $\mathit{Log}(FC)$, using Jankov and subframe formulas (the theories of these types of formulas for bi-GD were developed in [2, 9]). Since, by definition, this logic has the finite model property, we

[1] From now on we will use *extension* as a synonym of *axiomatic extension*.

can conclude that the problem of determining if a recursively axiomatizable extension of bi-GD is locally finite is decidable.

In this talk, we will cover the main steps of our recent proof. Namely, we will provide a characterization of the bi-Esakia duals of the finitely generated subdirectly irreducible algebras which validate bi-GD plus three particular Jankov formulas and one subframe formula. We will then present a combinatorial method we developed which can be used to show that the variety generated by the aforementioned algebras has the finite model property. This allows us to infer that $Log(FC)$ coincides with the extension of bi-GD axiomatized by the above mentioned Jankov and subframe formulas.

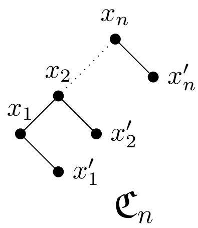

Figure 1: The n-comb $\mathfrak{C}_n$.

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Categorical Foundations for Fundamental Logic

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Holliday [1] recently introduced a non-classical logic called *Fundamental Logic*, which intends to capture exactly those properties of the connectives $\wedge, \vee$ and $\neg$ that hold in virtue of their introduction and elimination rules in Fitch's natural deduction system for propositional logic. Holliday provides an intuitive semantics for fundamental logic in terms of *fundamental frames* (sets endowed with a relation of openness between its points satisfying some conditions) which generalizes both Goldblatt's semantics for orthologic and Kripke semantics for intuitionistic logic.

The main goal of this talk (based on [3, Chap. 4] and [4]) is to provide some robust categorical foundations for Holliday's semantics for Fundamental Logic. First, we will show how his semantics naturally arises as the discretization of a duality between fundamental lattices (the natural algebraic companions of Fundamental Logic) and a subcategory of the category of Priestley spaces. The main construction, which is of independent technical interest, consists in using Priestley's duality between distributive lattices and Priestley spaces to a obtain a duality between the category of all lattices and a category of binary products of Priestley spaces.

Time permitting, we will also discuss how one can construct natural functors between the category of fundamental lattices and a category of fundamental frames, so as to obtain a version of the Goldblatt-Thomason theorem both for Fundamental Logic and for its modal extension [2].

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Maximal sublattices of convex geometries

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The origin of convex geometries lies in combinatorics, and the goal of the study of finite convex geometries was to develop the combinatorial abstraction of convexity. A convex geometry $G = (X, \alpha)$ is a finite closure system which satisfies the *anti-exchange property*, namely for all $x \neq y$ and all closed sets $A \in \mathcal{F} \subseteq 2^X$:

$$x \in \alpha(A \cup \{y\}) \textit{ and } x \notin A \textit{ imply that } y \notin \alpha(A \cup \{x\}).$$

The dual of a convex-geometry is an *antimatroid*: the family of complements of closed sets in G.

The closure lattices of a convex geometry have also been studied from the lattice theoretic point of view [3, 5]. A lattice L is (isomorphic to) the closure lattice of some convex geometry (shortly, we say: L is a convex geometry) iff, L is both:

- *join-semidistributive*: for every $x, y, z \in L$, $x \vee y = x \vee z$ implies that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, and
- *lower-semimodular*: for every $x, y \in L$, the covering relation $x \prec x \vee y$ implies $x \wedge y \prec y$.

In the nineties, a series of papers studied maximal sublattices and Frattini sublattices (intersection of all maximal sublattices), and considerable progress was done for lattices in classes $\mathcal{D}$ of distributive lattices and $\mathcal{B}$ of McKenzie's bounded lattices as shown in [1, 2]. But not much was known about classes that extends $\mathcal{D}$ and $\mathcal{B}$, as e.g. $\mathcal{CG}$ of (finite) convex geometries and $\mathcal{SD}_\vee$ of join semi-distributive lattices.

In this talk we show some results about maximal sublattices and Frattini sublattices in these two classes. In particular, there is a full description of maximal sublattices in convex geometries of convex dimension 2. The complements of maximal sublattices are precisely order-convex sets of one of the three forms (and satisfying in each case additional technical conditions):

- A singleton, namely a doubly-irreducible element.
- A chain.
- A union of two chains with the common least element.

Further, we present the conditions which have to be kept in order to obtain particular features of the Frattini sublattices. It is worth mentioning that convex geometries of convex dimension 2 are structures dual to SPS lattices, see e. g. [4].

This is a joint work with **K. Adaricheva** and **S. Silberger** from Hofstra University, as well as with **A. Zamojska-Dzienio** from Warsaw University of Technology.

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Three Theorems on Idempotent Semifields

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We call an algebraic structure $\langle S, \vee, \cdot, \mathrm{e}\rangle$ an *idempotent semiring* if

(i) $\langle S, \cdot, \mathrm{e}\rangle$ is a monoid;

(ii) $\langle S, \vee\rangle$ is a semilattice (i.e., an idempotent commutative semigroup); and

(iii) $a(b \vee c)d = abd \vee acd$ for all $a, b, c, d \in S$,

and an *idempotent semifield* if, additionally, $\langle S, \cdot, \mathrm{e}\rangle$ is the monoid reduct of a group. These structures play an important role in many areas of mathematics, including idempotent analysis, tropical geometry, formal language theory, and mathematical logic (see [6] for details). Other definitions of an idempotent semiring (also known as a *dioid* or an *ai-semiring*) may be found in the literature — in particular, an idempotent semifield may be defined without e in the signature, or with an extra constant symbol 0 interpreted as the neutral element of $\vee$, where $\langle S\backslash\{0\}, \cdot, \mathrm{e}\rangle$ is a group — but our results extend also to these settings.

Expanding an idempotent semifield $\langle S, \vee, \cdot, \mathrm{e}\rangle$ with the group inverse operation $^{-1}$ and lattice meet operation $\wedge$ defined by $a \wedge b := (a^{-1} \vee b^{-1})^{-1}$ produces a *lattice-ordered group* (or *ℓ-group*). Moreover, idempotent semifields are precisely the semiring reducts of ℓ-groups. In this work, which is developed in full in [8], we answer three open problems about equational theories of classes of idempotent semifields. These problems have been solved for classes of ℓ-groups, but restricting to fewer operations requires new proof methods and yields notably different results.

Let K be any class of $\mathcal{L}$-algebras for some signature $\mathcal{L}$, and call it *non-trivial* if at least one of its members is non-trivial, i.e., has more than one element. The *equational theory* $\mathrm{Eq}(\mathsf{K})$ of K is the set of all $\mathcal{L}$-equations $s \approx t$ such that $\mathsf{K} \models s \approx t$. A *basis* for this equational theory is a set of equations $\Sigma \subseteq \mathrm{Eq}(\mathsf{K})$ such that every equation in $\mathrm{Eq}(\mathsf{K})$ is a logical consequence of Σ. If $\mathrm{Eq}(\mathsf{K})$ has a finite basis, then K is said to be *finitely based*. Our first theorem is a complete answer to the finite basis problem for idempotent semifields. Although countably infinitely many equational theories of ℓ-groups have a finite basis (see, e.g., [2]), we prove, extending previous results obtained in [1], that:

Theorem A. *There is no non-trivial class of idempotent semifields that is finitely based.*

Our second theorem concerns the number of equational theories of classes of idempotent semifields. Using a technique of 'inverse elimination' introduced in [3, Section 4] to translate between equations in the different signatures, we obtain a one-to-one correspondence between a family of equational theories of ℓ-groups that is known to be uncountable (see [7]) and equational theories of certain classes of idempotent semifields, thereby proving:

Theorem B. *There are continuum-many equational theories of classes of idempotent semifields.*

The final theorem concerns the complexity of deciding equations in the class of idempotent semifields. The equational theory of the class of ℓ-groups is known to be co-NP-complete [5, Theorem 8.3] and we prove that this is also the case for the restricted signature, that is:

Theorem C. *The equational theory of the class of idempotent semifields is co-NP-complete.*

Using this result together with [4, Theorem 2], which relates the validity of equations in ℓ-groups to the existence of right orders on free groups, we also obtain the following:

Corollary. *Let* $\mathbf{F}(X)$ *be the free group over a set* X *with* $|X| \geq 2$*. Then the problem of checking for* $s_1, \ldots, s_n \in \mathrm{F}(X)$ *if there exists a right order* $\leq$ *on* $\mathbf{F}(X)$ *satisfying* $\mathrm{e} < s_1, \ldots, \mathrm{e} < s_n$ *is NP-complete.*

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Notes on ω-well-filtered spaces

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Abstract

We prove that a T_0 topological space is ω-well-filtered if and only if it does not admit either the natural numbers with the cofinite topology or with the Scott topology as its Skula closed subsets. Based on this, we offer a refined topological characterization for the ω-well-filterification of T_0-spaces and solve a problem of Xu.

In Mathematics, various types of "completions" of different structures have been drawing extensive attention and acting as important roles both in theory and practice. Examples include the Dedekind-MacNeille completion of ordered sets, completion of metrics and compactifications of topological spaces, et cetera. In domain theory and non-Hausdorff topology, completions such as the D-completion [6], well-filterification [9] and sobrification [3] of T_0 spaces are particularly well-studied in the form of reflectivity of the corresponding categories, to name a few.

The notion of ω-well-filtered spaces, which is strictly weaker than that of well-filtered spaces (hence that of sober spaces) introduced by Heckmann [5], is initially put forward by Xu et al. [10].

Definition 0.1. A T_0 space X is called *ω-well-filtered* if for every reversely ordered countably family $\{K_i\}_{i\in\mathbb{N}}$ of compact saturated subsets ($K_i \subseteq K_j$ when $i \geq j$), that the intersection $\bigcap_{i\in\mathbb{N}} K_i$ is in some open subset U implies that $K_i \subseteq U$ for some $i \in \mathbb{N}$.

Examples of ω-well-filtered spaces include all well-filtered spaces hence all sober spaces. Like well-filtered spaces and sober spaces, ω-well-filtered spaces have many nice properties. For example, the classical result that a sober space is locally compact if and only if it is core-compact can be extended to ω-well-filtered spaces. Xu et al. [10] showed that the category of all ω-well-filtered spaces is a reflective full subcategory of the category of T_0 spaces with continuous maps. This reveals the existence of "ω-well-filterification" for T_0 spaces. In the same paper, they gave a direct characterization of this completion by identifying the corresponding completion space as the family of all WD_ω-subsets endowed with the lower Vietoris topology.

In this paper, we look more closely at ω-well-filtered spaces through the lens of descriptive set theory. Recently, de Brecht obtained that a countably based T_0 space is sober if and only if it does not contain a Π^0_2-subspace homeomorphic to one of two specific topological spaces S_1 or S_D [2], where S_1 and S_D are the natural numbers with the co-finite topology and the Scott topology (in the usual order), respectively. This result was generalized to first-countable T_0 spaces in [7], and the authors showed that a first-countable T_0 space is sober if and only if it does not contain a Π^0_2-subspace homeomorphic to S_1, S_D or a directed subset without a maximum element. In a similar but different vein, and as a central result of this paper we prove:

Theorem 0.2. *A T_0 space is ω-well-filtered if and only if it does not contain S_1 or S_D as its Skula closed subsets.*

This provides characterizations for ω-well-filtered spaces by forbidden subspaces. In [10], Xu et al. also proved that on each first-countable T_0 space, well-filteredness and sobriety coincide. A natural question related to this matter is whether the well-filterification and sobrification constructions also coincide on first-countable T_0 spaces, which boils down to proving whether every first-countable T_0 space is a well-filtered determined space in the sense of Xu et al. [11]. The authors of [11] further posed the following problem:

Problem 0.3. Is every first-countable T_0 space a Rudin space?

Based on our aforementioned characterization for ω-well-filtered spaces, we solve Problem 0.3 in the negative by displaying a counterexample.

Moreover, our characterization for ω-well-filtered spaces via forbidden subspaces enables us to give more refined characterizations for the D-completion of Keimel and Lawson and also for the sobrification, when the underlying space is second-countable. This is achieved via the aid of a weaker version of the strong/Skula topology, which we introduce in this paper.

Definition 0.4 (Strong$_*$ topology)**.** Let X be a T_0 space. A nonempty subset A of X is said to have the *KF_ω property*, if there exists a countable filtered family $\mathcal{K}$ of compact saturated sets of X such that $\mathrm{cl}(A)$ is a minimal closed set that intersects all members of $\mathcal{K}$.
Let $\mathcal{B} = \{A \subseteq X \mid \sup B \in A \text{ for all } B \in \mathrm{KF}_\omega(A) \text{ with } \sup B \text{ existing}\}$. Then the family $\mathcal{B}$, as closed sets, forms the *strong$_*$ topology* of X.

Theorem 0.5. *1. In each ω-well-filtered space, all of its ω-well-filtered subspaces are precisely its closed subsets in the Strong$_*$ topology.*

2. The ω-well-filterification of a T_0-space X is homeomorphic to the Strong$_*$ closure of the embedding copy of X in the sobrification of X.

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When are bounded arity polynomials enough?

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This work is within the context of the classical Galois connection between sets of functions and sets of relations determined by the notion of *preservation*. For a set A, a function $f : A^n \to A$, and a relation $R \subseteq A^k$, we say that f *preserves* R (and write $f \rhd R$) whenever f applied coordinatewise to n-many tuples belonging to R produces another tuple belonging to R. For a finite set A, the Galois closed sets of functions are the *function clones*, while the Galois closed sets of relations are *relational clones*, which are sets of relations that are closed under positive primitive definitions. A classical and easy result in Universal Algebra states, for an equivalence relation $\theta \subseteq A^2$ and a function $f : A^n \to A$, that

$$f \rhd \theta \iff \mathsf{trl}_1(f) \rhd \theta,$$

where $\mathsf{trl}_1(f)$ is the set of all *basic translations* of f, i.e. those unary polynomials that can be produced by evaluating all arguments except possibly one at a constant. If a relation R satisfies the above property, we will write $\Xi_1(R)$.

Before this collaboration, each author had found a different kind of generalization of this result. In [1], the second author and his collaborators were interested in describing all relations R such that $\Xi_1(R)$ holds. Since $\Xi_1(\theta)$ holds for any quasiorder θ (reflexive and transitive binary relation), their work is focused on establishing properties of what they call *generalized quasiorders*, which are relations $R \subseteq A^k$ that are reflexive (contain the constant tuple $(c, \dots, c)$ for all $c \in A$) and transitive, which we now define. For $a \in A^{k^2}$, we will write $R \models a$ to indicate that every row and column of a when considered as a $k \times k$ matrix is a tuple belonging to R. With this notation, we say that R is transitive if whenever $R \models a$, then the diagonal of a is also an element of R. It is easy to see that this definition of transitivity coincides with the usual definition for binary relations. The authors establish that $\Xi_1(R)$ holds for any generalized quasiorder R.

On the other hand, in [2] the first author shows that properties of a higher arity commutator operation are closely connected to the properties of certain invariant relations called *higher dimensional equivalence relations*. Such relations are naturally coordinatized by higher dimensional cubes and satisfy natural generalizations of the transitive, reflexive, and symmetric properties ordinarily associated with binary relations. Higher dimensional congruences enjoy some nice properties, one of which is a generalization of the above equivalence to the following:

$$f \rhd \theta \iff \mathsf{trl}_d(f) \rhd \theta,$$

for a higher dimensional congruence $\theta \subseteq A^{2^d}$ and a function $f : A^k \to A$, where now we take $\mathsf{trl}_d(f)$ to be the set of polynomial functions obtainable from f by evaluating all but up to d-many variables at a constant. If a relation R satisfies this generalization of $\Xi_1(R)$, we will write $\Xi_d(R)$.

These two lines of inquiry prompt the search for a characterization of the those relations R for which $\Xi_d(R)$ holds. This question is closely related to a question about clones: for $M \subseteq A^{A^d}$

a set of d-ary operations on a finite set A, when is it true that

$$M^* = \{f \in \mathrm{Op}(A) : \mathsf{trl}_d(f) \subseteq M\}$$

is a clone? Among the results of this inquiry are definitions of reflexivity and transitivity which, on the one hand are suitable for a very broad class of relations, and on the other hand are each a natural generalization of the older concept. We are able to show that a relation R which is reflexive and transitive in the more general sense satisfies $\Xi_d(R)$ (for d a dimension parameter which we will not define here). Furthermore, we characterize those clones that are the polymorphisms of a set of such relations, for a particular dimension d. One of our conclusions is that each relation R with the property $\Xi_d(R)$ has a positive primitive definition in a particular relation Γ_M that is both reflexive and transitive in our general sense.

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Meet-irreducible elements in the poset of all logics

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In [2, 4, 3] the so-called *poset of all logics* is introduced and compared with the lattice Var of interpretability types of varieties (see, e.g., [1]). Roughly speaking, a variety V is *interpretable into* a variety W when W is term-equivalent to some variety, whose reducts (in a smaller language) belong to V. The interpretability relation for logics can be defined analogously, although it requires some tools from abstract algebraic logic (see, e.g., [2]).

More precisely, a (logical) matrix $(\mathcal{A}, F)$ is said to be a *model* of a logic $\vdash$ when F is a deductive filter of $\vdash$ on $\mathcal{A}$. In addition, if the unique congruence of $\mathcal{A}$ that does not glue an element of F to one of $A - F$ is the identity, we say that $(\mathcal{A}, F)$ is *reduced.* A matrix is a *Suszko model* of a logic $\vdash$ if it is isomorphic to a subdirect product of reduced models of $\vdash$. We denote by $\mathrm{Mod}^{\equiv}(\vdash)$ the class of all the Suszko models of $\vdash$ [2].

Let $\vdash$ be a logic. The set of connectives of $\vdash$ will be denoted by $\mathcal{L}(\vdash)$ and the set of terms of $\vdash$ with countably many variables by $\mathcal{T}(\vdash)$. Given two logics $\vdash$ and $\vdash'$, we say that a map $\tau\colon \mathcal{L}(\vdash) \to \mathcal{T}(\vdash')$ is a *translation* when it sends n-ary connectives to n-ary terms. In this case, with every algebra $\mathcal{A}$ in the language of $\vdash'$ we can associate an algebra $\mathcal{A}^{\tau}$ in the language of $\vdash$ defined as follows:

$$\mathcal{A}^{\tau} := (A, \{\tau(f)^{\mathcal{A}} : f \in \mathcal{L}(\vdash)\}).$$

We say that $\vdash$ is *interpretable* into $\vdash'$, in symbols $\vdash \leqslant \vdash'$, when there exists a translation $\tau\colon \mathcal{L}(\vdash) \to \mathcal{T}(\vdash')$ such that

$$(\mathcal{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash') \text{ implies that } (\mathcal{A}^{\tau}, F) \in \mathrm{Mod}^{\equiv}(\vdash).$$

Two logics $\vdash$ and $\vdash'$ are said to be *equi-interpretable* when $\vdash \leqslant \vdash' \leqslant \vdash$. We denote the equivalence class of all the logics that are equi-interpretable with $\vdash$ by $[\![\vdash]\!]$. Note that $\leqslant$ is a preorder on the class of all logics. The *poset of all logics* Log is the corresponding poset, whose elements are precisely the classes $[\![\vdash]\!]$[1]. Given two logics $\vdash$ and $\vdash'$, we write $[\![\vdash]\!] \leqslant [\![\vdash']\!]$ iff $\vdash \leqslant \vdash'$.

In [2] it is shown that even if Log has infima of families indexed by arbitrarily large sets, it may lack binary suprema (this is possible because its universe is not a set). Infima in Log can be described as follows. The *non-indexed product* of a family of algebraic languages $\{\mathcal{L}_i \mid i \in I\}$ is the algebraic language $\bigotimes_{i\in I} \mathcal{L}_i$ whose n-ary symbols are of the form $(\varphi_i(\overline{x}))_{i\in I}$, where each $\varphi_i(\overline{x})$ is an n-ary term of $\mathcal{L}_i$. Moreover, the *non-indexed product* of a family $\{\mathcal{A}_i \mid i \in I\}$, where each $\mathcal{A}_i$ is a $\mathcal{L}_i$-algebra, is the $\bigotimes_{i\in I} \mathcal{L}_i$-algebra $\bigotimes_{i\in I} \mathcal{A}_i$, whose universe is $\prod_{i\in I} A_i$ and whose n-ary symbols $(\varphi_i(x_1, \dots, x_n))_{i\in I}$ are interpreted as follows:

$$(\varphi_i(x_1, \dots, x_n))_{i\in I}^{\bigotimes_{i\in I} \mathcal{A}_i}(\overline{a}_1, \dots, \overline{a}_n) := (\varphi_i^{\mathcal{A}_i}(\overline{a}_1(i), \dots, \overline{a}_n(i)))_{i\in I}.$$

Similarly, the *non-indexed product* of a family of matrices $\{(\mathcal{A}_i, F_i) \mid i \in I\}$ is the matrix $(\bigotimes_{i\in I} \mathcal{A}_i, \prod_{i\in I} F_i)$. Lastly, the *non-indexed product* of a family $\{\vdash_i \mid i \in I\}$ of logics is the logic $\bigotimes_{i\in I} \vdash_i$ in the language $\bigotimes_{i\in I} \mathcal{L}_i$ induced by the class of matrices $\bigotimes_{i\in I} \mathrm{Mod}^{\equiv}(\vdash_i)$. It

[1]Although strictly speaking the universe of Log is not a set (and, therefore, Log is not a poset in the traditional sense), our results on this structure can be effortlessly rephrased in ZFC (see, e.g., [2]).

turns out that $[\![\bigotimes_{i\in I} \vdash_i]\!]$ is the infimum of $\{[\![\vdash_i]\!] : i \in I\}$ in Log [2, Thm. 4.6].

The aforementioned description of infima allows us to introduce a notion of meet-irreducibility for arbitrary logics. More precisely, we say that a logic $\vdash$ is *meet-irreducible* when $[\![\vdash]\!]$ is a meet-irreducible element of Log, i.e., for every pair of logics $\vdash_1$ and $\vdash_2$,

$$[\![\vdash_1 \otimes \vdash_2]\!] = [\![\vdash]\!] \text{ implies that either } \vdash_1 \leqslant \vdash \text{ or } \vdash_2 \leqslant \vdash .$$

Our main result provides a sufficient condition for the meet-irreducibility of a given logic. We say that a model of a logic $\vdash$ is *trivial* when it is either of the form $(\mathbf{1}, \{1\})$ or $(\mathbf{1}, \varnothing)$, where $\mathbf{1}$ is the trivial $\mathcal{L}(\vdash)$-algebra. On the other hand, recall that a class of similar matrices $\mathbb{K}$ has the *joint embedding property* (JEP) if for every set X of nontrivial members of $\mathbb{K}$ there exists some $(\mathcal{A}, F) \in \mathbb{K}$ in which every member of X embeds.

Theorem 1. *Every logic with theorems $\vdash$ satisfying the following conditions is meet-irreducible:*

(1) $\mathrm{Mod}^{\equiv}(\vdash)$ *has the JEP;*

(2) The nontrivial members of $\mathrm{Mod}^{\equiv}(\vdash)$ *have substructures of prime cardinality;*

(3) The nontrivial members of $\mathrm{Mod}^{\equiv}(\vdash)$ *lack trivial substructures.*

As a consequence, every intermediate logic is meet-irreducible and so are some prominent modal logics such as the global consequence of the normal modal logic S4.

It is natural to compare the above result with a well-known sufficient condition for meet-primeness in the lattice of interpretability types of varieties which states that, if V is the variety generated by a nested countable union of varieties V_n, where each V_n is generated by a finite algebra of prime cardinality, then V is meet-prime in Var [1, Prop. 18]. During the talk we will also discuss a variant of this observation in the context of logics (as opposed to varieties).

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Finitary semantics and languages of λ-terms

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Abstract

Salvati introduced a semantic notion of recognizable language of λ-terms in cartesian closed categories. The seminal work of Hillebrand and Kanellakis induces a syntactic notion of regular language of λ-terms. We show that these two notions coincide for a large class of cartesian closed categories. This shows the robustness of the notion of regular language of λ-terms as well as the dual one of profinite λ-term.

This is joint work with Sam van Gool, Paul-André Melliès and Tito Nguyễn.

There is a growing connection between automata theory and the theory of λ-calculus. Indeed, the Church encoding shows that finite words and ranked trees are simply typed λ-terms. For instance, words over the alphabet $\Sigma = \{a, b\}$ correspond to λ-terms of type

$$\mathsf{Church}_\Sigma \quad := \quad \underbrace{(\mathtt{o} \Rightarrow \mathtt{o})}_{a \text{ transition}} \Rightarrow \underbrace{(\mathtt{o} \Rightarrow \mathtt{o})}_{b \text{ transition}} \Rightarrow \underbrace{\mathtt{o}}_{\text{initial state}} \Rightarrow \underbrace{\mathtt{o}}_{\text{output state}}$$

Moreover, their semantic interpretations in the cartesian closed category **FinSet** coincides with their behavior in finite deterministic automata. This semantic observation led Salvati to define the notion of **recognizable language** in [7] as any set of λ-terms of a given type A of the form

$$\{M \in \Lambda(A) \mid [\![M]\!]_Q \in F\} \qquad \text{for some finite set } Q \text{ and subset } F \subseteq [\![A]\!]_Q.$$

The recognizable languages of type Church_Σ are then exactly the regular languages of words, seen through the Church encoding. Moreover, Salvati has shown that, for any type A, languages of λ-terms of that type assemble into a Boolean algebra. This definition, using finite sets, extends to any cartesian closed category.

There is another, more syntactic link between automata theory and λ-calculus. A seminal result by Hillberand and Kanellakis [3] states that a set of finite words is a regular language if and only if its characteristic function is λ-definable, modulo a type-casting operation sending any $M \in \Lambda(A)$ to $M[B] \in \Lambda(A[B])$. This observation is at the heart of the implicit automata program started in [5], which shows an analogous correspondence between star-free languages and planar λ-terms.

This line of work yields another, more syntactic notion of regular language of λ-terms of type A, implicit in the work of Hillebrand and Kanellakis. A **syntactically regular language** of λ-terms of a given type A is any set of the form

$$\{M \in \Lambda(A) \mid R\ M[B] =_{\beta\eta} \mathsf{true}\} \qquad \text{for some type } B \text{ and } \lambda\text{-term } R \in \Lambda(A[B] \Rightarrow \mathsf{Bool})$$

where Bool is the type $\mathtt{o} \Rightarrow \mathtt{o} \Rightarrow \mathtt{o}$ and true is the first projection.

In [4], we show that, for a large class of sufficiently well-behaved cartesian closed categories, the associated recognizable languages are exactly the syntactically regular ones. More precisely:

Theorem 1 (§7 of [4])**.** *A language of λ-terms of type A is recognizable by a non-thin well-pointed locally finite cartesian closed category if and only if it is syntactically regular.*

Theorem 1 provides evidence that the notion of recognizable language of λ-terms is robust, and does not depend on the category of finite sets. Its proof relies on a new construction on cartesian closed categories called **squeezing**, which is inspired by normalization by evaluation.

In [2], we have introduced profinite λ-terms, using semantic interpretation in finite sets, which assemble into a cartesian closed category **ProLam**. Profinite λ-terms of type Church_Σ are exactly the profinite words, and they extend the correspondance coming from Stone duality with regular languages [6, 1] in the following way:

Theorem 2 (Proposition 3.4 of [2]). *The space of profinite λ-terms of type A is the Stone dual of the Boolean algebra of regular languages of λ-terms of type A.*

Dually, the combination of Theorem 1 with Theorem 2 shows that the space of profinite λ-terms, initially defined in the setting of semantic interpretation in finite sets, does not depent on that construction.

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Prenex normal form theorems in intuitionistic arithmetic and the effective topos

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The fact that every first-order formula has a prenex normal form in classical logic is known as *prenex normal form theorem*, recognized as one of the most widely used theorems in mathematical logic. However, this theorem does not generally hold for intuitionistic theories. There has been work on this matter for more than half a century.

As studied in [1] and [2], it has been revealed that certain variants of prenex normal form theorem in Heyting arithmetic **HA** are strongly related to *semi-classical axioms* such as Γ-**DNE** (double negation elimination restricted to a class Γ of formulas). Akama et al. [1] initially introduced two syntactically defined classes of formulas, written as E_n and U_n. They are equivalent to Σ_n and Π_n, respectively, in the standard sense over classical logic. Akama et al. then show that $\mathbf{HA} + \Pi_n \vee \Pi_n$-**DNE** proves the prenex normal form theorem $\mathbf{PNFT}(\mathrm{U}_n, \Pi_n)$ from U_n to Π_n. Similarly, $\mathbf{HA} + \Sigma_n$-$\mathbf{DNE} + \Pi_n \vee \Pi_n$-**DNE** ensures both $\mathbf{PNFT}(\mathrm{E}_n, \Sigma_n)$ and $\mathbf{PNFT}(\mathrm{U}_n, \Pi_n)$.

In contrast, the situation concerning $\mathbf{PNFT}(\mathrm{E}_n, \Sigma_n)$ alone is more subtle. Fujiwara and Kurahashi [2] gave negative evidence: they show that there is an E_1-formula φ_0 that is not equivalent to any Σ_1-formula over $\mathbf{HA} + \Sigma_1$-**DNE**. This indicates that $\mathbf{PNFT}(\mathrm{E}_1, \Sigma_1)$ does not hold over $\mathbf{HA} + \Sigma_1$-**DNE**. The proof in [2] relies on a syntactic argument using a non-classical axiom, Church's thesis.

The purpose of this talk is to provide a topos-theoretic account on the last negative result concerning prenex normal form theorems in Heyting arithmetic. If an elementary topos $\mathcal{E}$ has a natural number object (NNO), $\mathcal{E}$ can be regarded as a model of Heyting arithmetic according to the standard interpretation of first-order logic. For instance, the *effective topos* $\mathcal{E}ff$, which is a significant example in categorical realizability, satisfies $\mathbf{HA} + \Sigma_1$-**DNE** but does not satisfy Σ_2-**DNE**. As seen from this example, a topos is not a model of classical arithmetic in general. However, by using the concept of *local operator*, it can be always "classicalized".

Local operator (a.k.a. Lawvere-Tierney topology) is the most important tool for creating a new topos from a given one. As a matter of fact, each local operator j in a topos $\mathcal{E}$ corresponds precisely to a subtopos $\mathcal{E}_j$ of $\mathcal{E}$. The logic of $\mathcal{E}_j$ may be different from the logic of $\mathcal{E}$. A typical example is the *double negation operator* $\neg\neg$, which exists in every topos. It is important that the corresponding subtopos always models classical logic even in the case the original topos does not. For example, the associated subtopos $\mathcal{E}ff_{\neg\neg}$ of the effective topos $\mathcal{E}ff$ is categorically equivalent to the category **Set** of sets.

As an illustration of the relationship between prenex normal form theorem and a topos-theoretic structure, we show the following theorem.

Theorem 1. *Let φ be an arithmetical formula and $\mathcal{E}$ an elementary topos with NNO satisfying Σ_n-**DNE**. In addition, suppose that φ is true in $\mathcal{E}$, while not in the subtopos $\mathcal{E}_{\neg\neg}$ associated with the double negation operator. Then there is no Σ_{n+2}-formula equivalent to φ over $\mathbf{HA}$ + Σ_n-**DNE**.*

The proof is based on a topos-theoretic notion, *transparency*, introduced in [3]. Furthermore, we can find concrete examples within subtoposes of the effective topos $\mathcal{E}ff$ for this theorem.

As is well known in categorical realizability, for every Turing degree d, there is a corresponding local operator j_d in $\mathcal{E}ff$. In particular, we can consider the local operator $j_{\emptyset^{(n)}}$ for the n-th Turing jump $\emptyset^{(n)}$ of the empty set. The associated subtopos $\mathcal{E}ff_{j_{\emptyset^{(n)}}}$ satisfies $\mathbf{HA}+\Sigma_{n+1}$-$\mathbf{DNE}$.

Theorem 2. *Let n be an arbitrary natural number. Then there exists an* E_{n+1}*-formula φ_n such that it is true in $\mathcal{E}ff_{j_{\emptyset^{(n)}}}$ but not true in $(\mathcal{E}ff_{j_{\emptyset^{(n)}}})_{\neg\neg} \simeq \mathbf{Set}$.*

The above theorems provide an alternative proof for the theorem in [2] and generalize it.

Corollary 3. *Let n be an arbitrary natural number. For the* E_{n+1}*-formula φ_n in Theorem 2, there is no Σ_{n+3}-formula equivalent to φ_n over* $\mathbf{HA}+\Sigma_{n+1}$-$\mathbf{DNE}$.

This implies that $\mathbf{PNFT}(\mathrm{E}_{n+1}, \Sigma_{n+1})$ does not hold over $\mathbf{HA}+\Sigma_{n+1}$-$\mathbf{DNE}$.

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The Effective Topos May be Simple Unstable

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Recent work has uncovered a fascinating connection between model theory and category theory.

- Hyland-Pitts [1]: the Turing Degrees embed effectively into the poset of Lawvere-Tierney topologies in the Effective Topos.
- Malliaris-Shelah [4]: the Turing Degrees embed effectively into Keisler's Order on simple unstable theories.

Leveraging these perspectives against each other gives two different ways of thinking about algorithmic complexity, but also gives some interesting new clues about categorical logic and model theory. By [2], we know that the Lawvere-Tierney topologies of Eff correspond to so-called "bilayered Turing Degrees". This sets up the following problem:

Problem 1. Can we modify Malliaris-Shelah's construction to embed the bilayered Turing Degrees into Keisler's Order?

Discussion 2. Why should this interest the category theorist? A key structure theorem in topos theory says: every Grothendieck topos $\mathcal{E}$ classifies a geometric theory $\mathbb{T}_{\mathcal{E}}$, and subtoposes of $\mathcal{E}$ correspond to quotients of $\mathbb{T}_{\mathcal{E}}$. It's natural to ask if this picture extends to elementary toposes and their subtoposes, but the previous structure theorem makes use of the site representation of Grothendieck toposes in a crucial way – this is not available for elementary toposes in general (e.g. the Effective Topos). A positive solution to Problem 1 means: given any LT-topology j in Eff, we can associate to it a theory $j \mapsto \mathbb{T}_j$ such that $j \leq j'$ iff $\mathbb{T}_j \trianglelefteq \mathbb{T}_{j'}$ in Keisler's Order — without recourse to the usual site representation.

Discussion 3. Why should this interest the model theorist? It is currently not known what the smallest upper bound of the Turing Degree theories are in Keisler's Order (if it even exists).[1] A positive solution to Problem 1 brings into focus a potential new dividing line within simple unstable theories: [5, Prop 3] says that if j is a non-trivial LT topology in Eff such that $j_A \leq j$ for any Turing Degree topology j_A, then $j_{\neg\neg} = j$. This would also clarify the picture of how we might understand simple theories as being built out of certain basic building blocks, raising interesting implications for viewing Keisler's Order as a systematic search for important partition patterns of set systems.

This talk will introduce and clarify the connection between these two embeddings of the Turing Degrees, with a view towards Problem 1. As a baseline step towards its solution, we focus on how both the Effective Topos & Keisler's Order use topological ideas to calibrate jumps in complexity. Time permitting, we may discuss Lee-van Oosten's result that each LT-topology in Eff is built from a family of basic LT-topologies [3], and explore its ramifications.

[1] An upper bound exists since there exists a maximal class in Keisler's Order (which includes theories satisfying SOP_1); the question is how much further down can we move this upper bound. In particular, recall that there does not exist a maximal Turing Degree.

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NP-hardness of promise colouring graphs via homotopy

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The so-called algebraic approach to constraint satisfaction problems is well-established and successful example of application of universal algebra in computational complexity. This line of research started with a conjecture of Feder and Vardi [3], that each finite-template CSP is either in P or NP-complete. The algebraic theory of polymorphisms was established by Jeavons et al. [5, 1], and the approach culminated with two independent positive resolutions of the conjecture by Bulatov [2] and by Zhuk [8].

Fixed-template constraint satisfaction problems (CSPs) may be defined in several ways: The goal is to decide whether a given primitive positive formula is satisfiable in a fixed finite structure (called *template*). Alternatively, it is a homomorphism problem for finite relational structures where the target structure is fixed, i.e., we are asking given a structure X whether there is a homomorphism to a fixed structure A. The problem is usually denoted by $\mathrm{CSP}(A)$.

A substantial recent effort has been dedicated to a slight generalisation of CSPs to promise problems. A promise problem consists of two disjoint (but not necessarily complementary) sets of instances: positive and negative. A *promise CSP* is a promise problem whose positive instances are positive instance of $\mathrm{CSP}(A)$ and whose negative instances are negative instances of another $\mathrm{CSP}(B)$; note that in order for these to be disjoint, we have to have a homomorphism from A to B. A prototypical example of a promise CSP is approximate graph colouring: Given a graph G, decide between the case that G is 3-colourable and the case that G is not even 6-colourable. The algebraic methods generalise to the promise setting, but universal algebra is less relevant in resolving the complexity of these problems, and new tools need to be developed to address these problems.

The goal of this talk is to share one of these new tools. I will outline a new application of topology (more precisely, homotopy theory) in assessing the complexity of promise CSPs. Namely, I will talk about hardness of two versions of graph and hypergraph colouring:

- It is NP-complete to decide between graphs that (A) map homomorphically to an odd cycle and those that (B) are not 3-colourable (Krokhin, Opršal, Wrochna, and Živný [6]).
- It is NP-complete to decide between 3-uniform hypergraphs (A) that can be coloured by 3 colours in such a way that each edge has a unique maximum, and (B) those that cannot be coloured by 4 colours in the same way. (Filakovský, Nakajima, Opršal, Tasinato, and Wagner, [4]).

Both proofs are based on topological ideas first used to show lower bounds of the chromatic number of Kneser graphs by Lovász [7] in conjunction with algebraic and categorical tools.

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Polynomial time checking of generalized Sahlqvist shape

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In classical normal modal logic, the class of Sahlqvist formulae has several desirable properties such as defining canonical logics [15, 16]. The proof of canonicity of the logics defined by Sahlqvist axioms is obtained by proving that Sahlqvist formulae are elementary, i.e., the classes of frames they define are also defined by first order sentences. Besides the original Sahlqvist-van Benthem algorithm to compute such *first order correspondent* for any given Sahlqvist formula, other algorithms for second order quantifier elimination have been adapted to Sahlqvist formulas, such as SCAN [9] and DLS [8, 14]. Sahlqvist formulas and their correspondents are interesting also from a proof-theoretic perspective: for instance, Negri has shown that analytic calculi can be effectively generated for all the modal logics in the Sahlqvist fragment [13], since the first order correspondents of Sahlqvist formulas are generalized geometric formulas.

In [11], Goranko and Vakarelov extended Sahlqvist canonicity and correspondence results to the class of *inductive formulas* (also known as *generalized Sahlqvist* formulas), which is strictly large than the class of Sahlqvist formulas. Based on SCAN and DLS, the algorithm SQEMA for correspondence on inductive formulas has been introduced in [1, 3].

By reframing Sahlqvist theory in algebraic terms, the syntactic notion Sahlqvist and inductive formulas have been imported as Sahlqvist and inductive *inequalities* in much more general settings, and correspondence and canonicity properties analogous to the classical ones have been proved [7, 2, 4, 6]. Such developments extend Sahlqvist theory to all the logics the algebraic semantics of which are given by (distributive) normal lattice expansions (LE), e.g., intuitionistic modal logic, positive modal logic, orthologic, the full Lambek calculus, the multiplicative-additive fragment of linear logic, semi De-Morgan logic, and so on. The Ackermann Lemma Based Algorithm (ALBA) has been introduced as a successor of SQEMA to compute first order correspondents in such a general setting. Similarly, also proof theoretic results concerning inductive inequalities in such logics which partially extend the classical ones have been proved in [12], and results proving canonicity in a constructive meta-theory reflecting [10] have been proved in [5].

Contrary to the classical case, checking whether a given inequality is inductive is not an obviously easy task. Indeed, the strong properties characterising the Boolean setting make it possible to define the class of Sahlqvist inequalities in a way that straightforwardly induces a polynomial-time algorithm (on the length of the formula) to check whether a formula belongs in this class. The definition of inductive (and Sahlqvist) inequality in the more general LE setting is more involved, and a naive approach would check a certain property (in polynomial time) for each strict order on the variables, and for each polarity (either positive or negative) assignment on the variables; hence it would have time complexity $\mathcal{O}(2^v v! p(n))$, where v is the number of variables in the inequality, n is the length of the formula, and p is a polynomial.

In this talk, we show an algorithm that computes whether an inequality is *refined inductive* in polynomial time, i.e., $\mathcal{O}(n+vl+v^2h+h^2)$, where n is the length of the formula, l the number of leaves in its syntax tree, v the number of variables in the inequality, h the number of topmost nodes having a certain property in the syntax tree.

Checking whether an inequality is inductive is equivalent to checking for the existence of a system of *refined inductive* inequalities which is semantically equivalent (relative to the appropriate class of LEs) to the given inequality. Since the algorithm ALBA for correspondence (on which most of the applications of inductive inequalities rely) pre-processes any input inductive inequality so as to obtain such a system of refined inductive inequalities, this algorithm finds its natural place in a practical implementation of ALBA in the preprocessing step applied to any input inequality.

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Semigroups in Classical Planning

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Classical planning [3] has become a major paradigm in areas of applied computer science, such as robotics, logistics and manufacturing. Although the class of programs under its scope (classical plans) are strickingly simple, current research still depends on traditional formalisms that are largely disconnected from abstract mathematics. (An exception being recent attempts to subsume planning, combined with computer vision or reasoning, into category theory [1].)

To this end, we suggest an algebraic appropach to classical planning that: (1) brings this area closer to mathematical practice; and (2) permits an abstract approach to plans that benefits research in the area of classical planning itself.

In a nutshell, classical planning is the search for deterministic plans that lead to a goal from the initial state. Classical plans $\pi \in A^*$ are built from a set A of primitive actions available to an agent. Given a finite set of logical atoms $At = \{p, q, \ldots\}$ and literals $Lit = \{p, -p, q, \ldots\}$, a state $s \in S$ is a maximally consistent set of literals. The goal is just a consistent set of literals, and so are the preconditions and effects that define an action a as a pair $a = (pre(a), \mathit{eff}(a))$.

For action updates, one defines first a consistency-preserving update function over sets of literals X, Y

$$X \diamond Y = (X \setminus -Y) \cup Y$$

where $-Y := \{-y : y \in Y\}$. Action or plan executions are then defined by a function $\gamma : S \times A^* \to S$ (technically, a semigroup action) where:

$$\gamma(s, a) = \begin{cases} s \diamond \mathit{eff}(a) & \text{if } s \models pre(a) \\ \mathit{undefined} & \text{otherwise} \end{cases} \qquad \begin{array}{rcl} \gamma(s, \langle\rangle) & = & s \\ \gamma(s, a.\pi) & = & \gamma(\gamma(s, a), \pi) \end{array}$$

(Here, $\langle\rangle$ is the empty plan, and the plan $a.\pi$ is the concatenation of a and π.)

Let us now turn into algebra, by abstracting from the goal and initial state that define a planning problem. Henceforth, a plan is just a finite action sequence.

A semigroup $(G, \cdot)$ consists of an associative operation $\cdot : G \times G \to G$ on a set G. Two immediate semigroups capture the syntax and semantics of plans:

(1) the free (word) semigroup $(A^*, .)$ of plans π built under concatentation '.'

(2) the semigroup $(\|A^*\|, \circ)$ of plan executions $\|\pi\|$ under map composition $\circ$.

Each plan π does correspond to a partial transformation $\|\pi\| : S \to S$ given by $\|\pi\|(s) = \gamma(s, \pi)$. Indeed, (2) is a subsemigroup of $\mathcal{PT}(S)$, the semigroup of partial transformations of S, thoroughly studied in [2]. Semantically, there is thus no difference between actions and plans: they are just partial maps. To replicate this uniformity at the level of syntax, we define a product $\bullet : A \times A \to A$ that reduces plans to actions (for any set A closed under $\bullet$) so as to obtain:

(3) the semigroup $(A, \bullet)$ of actions $a = (pre(a), \mathit{eff}(a))$.

Let $\mathbf{A}_0$ contain all planning actions plus a zero 0, that we introduce with the *false* constant as the action $0 = (\{\perp\}, \{\perp\})$. (Note that $\gamma(s, 0) = \mathit{undefined}$ for any $s \in S$.) We define the product in (3) by:

$$a \bullet b = \begin{cases} (\mathit{pre}(a \bullet b), \mathit{eff}(a \bullet b)) & \text{if } \mathit{pre}(b) \cap -(\mathit{pre}(a) \diamond \mathit{eff}(a)) = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $\mathit{pre}(a \bullet b) = \mathit{pre}(a) \cup (\mathit{pre}(b) \setminus \mathit{eff}(a))$ and $\mathit{eff}(a \bullet b) = \mathit{eff}(a) \diamond \mathit{eff}(b)$.

After proving that (3) is a semigroup, we verify that its product $\bullet$ is correct:

$$\|a \bullet b\| = \|a.b\| = \|a\| \circ \|b\|.$$

Next we fully characterize in each semigroup (1)–(3): the identity 1, the zero 0 and invertible elements $a = a^{-1}$; the zero divisors $ax = 0 = xa$ (for some x), nilpotents $aa = 0$, idempotents $aa = a$ and their natural partial ordering ($a \leq b$ iff $ab = a = ba$); and also commutativity $ab = ba$.

A function $\mathit{can}(a) = (\mathit{pre}(a), \mathit{eff}(a) \setminus \mathit{pre}(a))$ further identifies non-redundant actions as canonical representatives of behaviourally equivalent actions, in the sense that $\|\mathit{can}(a)\| = \|a\|$. Such non-redundant actions arrange into:

(3') the semigroup $(\mathbf{A}'_0, \bullet')$ of canonical actions, where $a \bullet' b = \mathit{can}(a \bullet b)$

Then we prove an isomorphism ($\leftrightarrow$) between (3') and the partial transformations induced by constructible plans $(\|\mathbf{A}^*\|, \circ)$ (see left figure):

$$\begin{tikzcd} & \mathbf{A} \arrow[dl, dotted, "\bullet"'] \arrow[dr, dotted, "."] & \\ (\mathbf{A}_0, \bullet) \arrow[d, "\mathit{can}(\cdot)"'] & & (\mathbf{A}^*, .) \arrow[ll, "\bullet"] \arrow[d, "\|\cdot\|"] \\ (\mathbf{A}'_0, \bullet') \arrow[rr, leftrightarrow, "\|\cdot\|"] & & (\|\mathbf{A}^*\|, \circ) \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{L} \arrow[dr] & \\ \mathit{id} = \mathcal{H} \arrow[ur] \arrow[dr] & & \mathcal{D} = \mathcal{J} \\ & \mathcal{R} \arrow[ur] & \end{tikzcd}$$

Finally, (see right figure) we identify the Green relations and principal ideals in (3'):

$$\begin{array}{rcll} a\mathcal{L}b & \text{iff} & \mathbf{A}'_0\, a = \mathbf{A}'_0\, b & \text{iff } -(\mathit{pre}(a) \,\Delta\, \mathit{pre}(b)) \subseteq \mathit{eff}(a) = \mathit{eff}(b) \\ a\mathcal{R}b & \text{iff} & a\, \mathbf{A}'_0 = b\, \mathbf{A}'_0 & \text{iff } \mathit{pre}(a) = \mathit{pre}(b) \text{ and } \mathit{eff}(a) \,\Delta\, \mathit{eff}(b) = -(\mathit{eff}(a) \Delta \mathit{eff}(b)) \\ a\mathcal{H}b & \text{iff} & a(\mathcal{L} \cap \mathcal{R})b & \text{iff } a = b \\ \mathcal{D} & = & \text{min. equiv.} \supseteq \mathcal{L}, \mathcal{R} & \text{iff } -(\mathit{pre}(a) \cup \mathit{pre}(b)) \subseteq \mathit{eff}(a) \cap \mathit{eff}(b) \text{ and } \mathit{pre}(a) \cap \mathit{eff}(b) = \emptyset \\ a\mathcal{J}b & \text{iff} & \mathbf{A}'_0 a\, \mathbf{A}'_0 = \mathbf{A}'_0 b\, \mathbf{A}'_0 & \qquad \ldots \text{and } \mathit{eff}(a) \,\Delta\, \mathit{eff}(b) \subseteq -(\mathit{eff}(a) \,\Delta\, \mathit{eff}(b)). \end{array}$$

Our results offer a solid and elegant foundation to classical planning, with potential applications in the study of heuristic search functions, plan-space planning and partial-order planning, among other research lines in the area.

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The Category of Approximation Spaces*

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Approximation Fixpoint Theory (AFT) [5] is an algebraic framework designed to study the semantics of non-monotonic logics, like logic programming, autoepistemic logic, and default logic, and to resolve longstanding problems on the relation between these formalisms [6]. The core ideas of AFT are relatively simple: we are interested in fixpoints of an operator on a given lattice $\langle L, \leq \rangle$. For monotonic operators, Tarski's theory guarantees the existence of a least fixpoint. AFT generalizes Tarki's theory to non-monotonic operators by making use of a so-called *approximating operator*: an operator $A : L^2 \to L^2$ monotonic with respect to the precision order $\leq_p$ (defined by $(x, y) \leq_p (u, v)$ if $x \leq u$ and $v \leq y$)). The intuition is that elements of L^2 are used to approximate elements of L: the tuple $(x, y) \in L^2$ is said to approximate z if $x \leq z \leq y$. Given such an approximator, AFT defines several types of fixpoints (supported fixpoints, a Kripke-Kleene fixpoint, stable fixpoints, and a well-founded fixpoint) of interest.

Let us illustrate the application of AFT to standard, first-order, logic programming. In this setting, the lattice L is the lattice of interpretations, ordered by the truth order $I \leq J$ if $P^I \subseteq P^J$ for each predicate P. The operator at hand is the immediate consequence operator T_P of a logic program P [9]. In this setting, pairs (I, J) can be seen as *four-valued* interpretations: a fact q is true if it is *true* in both I and J, *false* if it is false in both I and J, *unknown* if it is true in J but not true in I and *inconsistent* if it is true in I but not in J. The approximating operator Ψ_P is, in this case, nothing more than Fitting's four-valued immediate consequence operator [7].

This research is motivated by a need to apply AFT to *higher-order* logic programming that arose in several contexts [3, 2, 8]. An important issue in this context is that using pairs of interpretations no longer allows for an obvious way to evaluate formulas in an approximation. Let us illustrate this with an example. Consider a logic program in which a first-order predicate p and a second-order predicate Q are defined. Now assume that in the body of a rule, the atom $Q(p)$ occurs. A tuple (I, J) of interpretations in this case tells us for any given set S if $Q(S)$ is true, false, unknown, or inconsistent. However, the interpretation of p in such an interpretation (I, J) is not a set, but a partially defined set, making it hard to evaluate expressions of the form $Q(p)$. To deal with definitions of higher-order objects, approximate interpretations should take into account the application of approximate objects to approximate objects. This suggests that spaces of approximations of higher-order objects should be defined inductively from lower-order ones, following the type hierarchy: we start by assigning a *base approximation space* to each type at the bottom of the hierarchy, and then, for each composite type $\tau_1 \to \tau_2$, we define its approximation space as *a certain class of functions* from the approximation space for τ_1 to the approximation space for τ_2. The main question is how to define the base approximation spaces and the class of functions in a generic way that works in all applications of AFT.

Clearly, we want to be able to apply the same AFT techniques at any level of the hierarchy, i.e. all approximation spaces should share the same algebraic structure. In Category Theory (CT), there already exists a notion that captures this behavior: the concept of *Cartesian closed*

*This research has been conducted under the supervision of Bart Bogaerts (Vrije Universiteit Brussel, Brussels, Belgium), and Marc Denecker (KU Leuven, Leuven, Belgium).

category (ccc). The objects of a ccc **C** satisfy a property that can be intuitively understood as follows: *if A and B are two objects of* **C***, then the set of morphisms from A to B is also an object of* **C**. It follows that, if the base approximation spaces are objects of a ccc, then the category also contains the full hierarchy of approximation spaces we are aiming for. We will call such a ccc an *approximation category* and denote it by **Approx**. Clearly, the definition of **Approx** depends on the application we want to use AFT for. Different applications imply different higher-order languages, with different types, and possibly different versions of AFT (standard AFT [5], consistent AFT [4], or other extensions [1]). To formalize this, we develop the notion of an *approximation system*. Once a language and the semantics of its types are fixed, we can choose an approximation system that consists, among other things, of a ccc **Approx**, equipped with a function *App* associating the semantics of a type to an approximation space in **Approx**. The approximation system also determines which elements of the approximation spaces are *exact*, i.e. which elements approximate exactly one element of the semantics of a type, and, for every type, it provides a projection from the exact elements to the objects they represent in the corresponding semantics. This is non-trivial for higher-order approximation spaces, and it is indeed fundamental to obtain a sensible account for AFT for higher-order definitions. Thanks to the generality of this formalization, there are several viable choices for an approximation system. For instance, we show that the bilattices form a ccc with the monotone functions as morphisms. With a suitable choice of *App* and exact elements we obtain an approximation system that recovers the framework of standard AFT and extends it to higher-order objects. Furthermore, we have shown that the approximation spaces from [1] form a ccc. Our approach provides a clear definition for exact higher-order elements, which was missing in the work [1].

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A Stone duality for the class of compact Hausdorff spaces

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The Stone space of a boolean algebra is defined as the totally disconnected compact Hausdorff space whose points are the ultrafilters of the boolean algebra. Conversely, a totally disconnected compact Hausdorff space is homeomorphic to the Stone space of the boolean algebra of its clopen sets. Looking from a categorical perspective, the Stone duality can be expressed as the existence of a duality between the category of boolean algebras with arrows represented by adjoint homomorphisms and the category of totally disconnected compact Hausdorff spaces with continuous open maps. Furthermore, this duality restricts to one between the category of complete boolean algebras with complete homomorphisms and the category of extremally disconnected compact Hausdorff spaces with continuous open maps.
Generalizing this idea, we present an algebraic characterization of T_0-topological spaces in terms of preorders describing a base for the space. In particular, we show that any T_0-topological space can be represented as the space whose points are the neighborhood filters of one of its basis for the open sets. Conversely, we show that any dense family of filters on a preorder defines a topological space whose characteristics are strictly connected to the ones of the preorder. Therefore, we show how the separation properties of the topological space can be described in terms of the algebraic properties of the corresponding preorder and family of filters.
Furthermore, drawing on Orrin Frink's article [1], we outline the algebraic conditions on a selected base of the topological space ensuring that the space is compact and Hausdorff.
Indeed, in his article [1], Frink provided an internal characterization of Tychonoff spaces: specifically, he proved that a space is Tychonoff if and only if it admits a *normal* base for the closed sets of a space, i.e. a base that forms a disjoint ring of sets, where disjoint members can be separated by disjoint complements of members of the base. Moreover, he showed that if Z is a normal base for X, then the space of Z*-ultrafilters* forms a Hausdorff compactification for X. Clearly, we can obtain different compactifications of the same non-compact Tychonoff space by choosing different normal bases.
Following this approach, we characterize the algebraic properties that a preorder must possess in order to induce a Tychonoff space. In particular, we show that every Tychonoff space can be described as the space whose points are *some* minimal prime filters of a particular type of distributive lattices. Furthermore, we show that the space obtained considering *all* the prime minimal filters of it forms a Hausdorff compactification of the original Tychonoff space.
These results allow us to define a duality between the category of compact Hausdorff spaces with continuous maps and a suitable category of lattices.
This is joint work with Matteo Viale.

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Equivalential Algebras With Conjunction on Dense Elements

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Abstract

We study the variety generated by the three-element equivalential algebra with conjunction on the dense elements. We prove the representation theorem which let us construct the free algebras in this variety. Next, we compute the formula for the cardinality of these algebras.

1 Introduction

According to [2], there are only finitely many polynomial clones on a finite algebra which generate congruence permutable Fregean varieties. A variety $\mathcal{V}$ with a distinguished constant term 1 is called **Fregean** if every algebra $\mathbf{A} \in \mathcal{V}$ is: **1-regular**, (i. e., $1/\alpha = 1/\beta$ implies $\alpha = \beta$ for all $\alpha, \beta \in \operatorname{Con}\mathbf{A}$) and **congruence orderable** (i. e., $\Theta_{\mathbf{A}}(1, a) = \Theta_{\mathbf{A}}(1, b)$ implies $a = b$ for all $a, b \in A$) [2, p. 597].

If a three-element algebra $\mathbf{A}$ generates a congruence permutable Fregean variety, then the lattice of congruences on $\mathbf{A}$ is a three-element chain. By [2, Corollary 2.8], due to the behavior of the commutator operation on a three-element algebra, we can distinguish four polynomially nonequivalent algebras, that generate congruence permutable Fregean varieties.

Two of them are well known: the three-element equivalential algebra and the three-element Brouwerian semilattice. The equivalential algebras are solvable, so they are of type 2 ([2, p. 606]) in the sense of Tame Congruence Theory of Hobby and McKenzie [1]. However, the Brouwerian semilattices are congruence distributive and so they are of type 3.

In the other two cases we are dealing with a mixed type. In the first case, we have type 3 at the top of congruence lattice and type 2 at its bottom. An example of algebra, which meets these conditions is the three-element equivalential algebra with conjunction on the regular elements. The variety generated by this algebra was investigated in [3].

2 Main results

The aim of this talk is present recent results on the variety generated by the three-element algebra, in which the commutator operation behaves in the opposite way: type 2 is at the top of congruence lattice and type 3 at its bottom (most of these results can be found in the article [4], written with Katarzyna Słomczyńska). Such structure is the subreduct of the three-element Heyting algebra, with the equivalence operation and the second binary operation which is conjunction on the dense elements.

Definition 1. An **equivalential algebra with conjunction on the dense elements** is an algebra $\mathbf{D} := (\{0, *, 1\}, \cdot, d, 1)$ of type $(2, 2, 0)$, which is the reduct of the three-element Heyting algebra $\mathbf{H} = (\{0, *, 1\}, \wedge, \vee, \rightarrow, 0, 1)$ with an order: $0 < * < 1$, the constant 1, the equivalence operation $\cdot$ such that $x \cdot y := (x \rightarrow y) \wedge (y \rightarrow x)$ (we adopt the convention of associating to the left and ignoring the symbol of equivalence operation), and an additional binary operation d such that: $d(x, y) := x00x \wedge y00y$.

We denote by $\mathcal{V}(\mathbf{D})$ the variety generated by $\mathbf{D}$. A crucial role in the construction of the finitely generated free algebras is played by the subdirectly irreducible algebras in $\mathcal{V}(\mathbf{D})$.

Proposition 2. *There are only three (up to isomorphism) nontrivial subdirectly irreducible algebras in* $\mathcal{V}(\mathbf{D}) : \mathbf{D}, \mathbf{2}, \mathbf{2}^{\wedge}$, *where:*

$$\mathbf{2} := \{\{0,1\}, \cdot, d, 1\}, \text{ where } d \equiv 1,$$

$$\mathbf{2}^{\wedge} := \{\{*,1\}, \cdot, d, 1\}, \text{ where } d(x,y) := x \wedge y.$$

Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$. We denote by $\mathrm{Cm}(\mathbf{A})$ the set of all completely meet-irreducible congruences on $\mathbf{A}$ and we use the following notation:

$$\begin{aligned} \overline{L} &:= \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}\}, \\ \underline{L} &:= \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{D}\}, \\ P &:= \{\mu \in \mathrm{Cm}(\mathbf{A}) : \mathbf{A}/\mu \cong \mathbf{2}^{\wedge}\}, \\ L &:= \overline{L} \cup \underline{L}. \end{aligned}$$

To construct the free algebras in $\mathcal{V}(\mathbf{D})$ we need the notion of the hereditary sets.

Definition 3. Let $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ and $Z \subseteq \mathrm{Cm}(\mathbf{A})$. A set Z is **hereditary** if:

1. $Z = Z\uparrow$,
2. $\overline{L} \subseteq Z$ or $((\overline{L} \cap Z) \cup \{\mathbf{1}_{\mathbf{A}}\}, \bullet)$ is a hyperplane in $(\overline{L} \cup \{\mathbf{1}_{\mathbf{A}}\}, \bullet)$, where $\mu_1 \bullet \mu_2 := (\mu_1 \div \mu_2)'$ for $\mu_1, \mu_2 \in \overline{L}$ ($\div$ denotes the symmetric difference)

We will denote by $\mathcal{H}(\mathbf{A})$ the set of all hereditary subsets of $\mathrm{Cm}(\mathbf{A})$.

Now we give our main result, i.e. the representation theorem:

Theorem 4. *Let* $\mathbf{A} \in \mathcal{V}(\mathbf{D})$ *and let* $\mathbf{A}$ *be finite. Then the map* $M : A \ni a \to M(a) := \{\mu \in \mathrm{Cm}(\mathbf{A}) : a \in 1/\mu\}$ *is the isomorphism between* $\mathbf{A}$ *and* $(\mathcal{H}(\mathbf{A}), \leftrightarrow, \mathsf{d}, \mathit{1})$, *where*

$$\begin{aligned} Z \leftrightarrow Y &:= ((Z \div Y)\downarrow)' \\ \mathsf{d}(Z,Y) &:= [Z \cup ((Z\downarrow)' \cap L)] \cap [Y \cup ((Y\downarrow)' \cap L)], \\ \mathit{1} &:= \mathrm{Cm}(A), \end{aligned}$$

for $Z, Y \in \mathcal{H}(\mathbf{A})$.

Using this theorem, we can construct the finitely generated free algebras in $\mathcal{V}(\mathbf{D})$, which we denote by $\mathbf{F}_{\mathbf{D}}(n)$, and find the formula for the cardinality of these algebras:

$$|\mathbf{F}_{\mathbf{D}}(n)| = 2^{3^n - 2^n} + \sum_{k=1}^{n} \binom{n}{k} 2^{\frac{3^n + 3^{n-k}}{2} - 2^{n-1}}.$$

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The algebras of Lewis's counterfactuals and their duality theory

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A counterfactual conditional (or simply a counterfactual) is a conditional statement of the form "If *antecedent* were the case, then *consequent* would be the case", formalized as "$\varphi \Box\!\!\rightarrow \psi$" where the antecedent is usually assumed to be false. Counterfactuals have been studied in different fields, such as linguistics, artificial intelligence, and philosophy. The logical analysis of counterfactuals is rooted in the work of Lewis [4, 3] and Stalnaker [8] who have introduced what has become the standard semantics for counterfactual conditionals based on particular Kripke models (called *sphere models*) equipped with a similarity relation among the possible worlds. Lewis [4] develops a hierarchy of logics meant to deal with different kinds of counterfactual conditionals; they are usually referred to as *variably strict conditional logics*.

Although the research on Lewis's conditional logics has been and still is very prolific, the algebraic perspective is essentially lacking; while a few works present a semantics in terms of algebraic structures for Lewis's conditional logics ([5, 7]), the results therein are either partial or fall outside the framework of the abstract algebraic analysis. A foundational work that carries Lewis's hierarchy within the realm of the well-developed discipline of (abstract) algebraic logic is notably missing in the literature; the present contribution aims at filling this void.

To this end, we start by considering Lewis's logics as consequence relations, instead of just sets of theorems, and we introduce novel (and simpler) axiomatizations. This brings us to consider two different kinds of derivation, depending on whether the deductive rules are applied only to theorems (giving a relatively weaker calculus) or to all derivations (i.e. yielding a stronger calculus); this distinction, although relevant, is often blurred in the literature. As it is the case for modal logic (see [1, 9]), these two choices turn out to correspond to considering two different consequence relations on the intended sphere models: a *local* and a *global* one; the latter, to the best of our knowledge, has not been considered in the literature.

Inspired by some results connecting modal operators and Lewis counterfactuals (see [4]), our work unveils a deep relationship between Lewis's logics and modal logic. Specifically, we demonstrate how several model-theoretic techniques commonly used in standard Kripke semantics for modal logic (such as the generated sub-model construction) can be successfully applied to Lewis's sphere semantics, thanks to a modal operator $\Box$ that can be term-defined in the language. This allows us to, for example, prove a deduction theorem for the strong calculus (whereas the weak calculus is known to have the classical deduction theorem) and to characterize the global consequence relation in terms of the local one, paralleling analogous well-known results in modal logic (see [9]).

Furthermore, we introduce a new variety of algebras, that we call V-algebras, consisting of Boolean algebras equipped with a binary operator $\Box\!\!\rightarrow$ that stands for the counterfactual conditional. We show that the stronger calculi, associated to the global consequence relation, are strongly algebraizable in the sense of Blok-Pigozzi, with respect to (subvarieties of) V-algebras. In turn, we demonstrate that the weaker calculi, associated to the local consequence relation, are not algebraizable in general, but they correspond to the logics *preserving the degrees of truth* of the same algebraic models. Thus the same class of algebras can be meaningfully used to study both versions of Lewis's logics; precisely, we have strong completeness of both calculi with respect to V-algebras. We also initiate the study of the structure theory of the algebraic models; interestingly, we demonstrate that the congruences of the algebras, which are in one-one correspondence with the deductive filters inherited by the logics, can be characterized by means of the congruences of their modal reducts.

The second part of our work develops a duality result for V-algebras, circling back to the original intended sphere models. In more details, we show two different dual categorical equivalences of our algebraic structures with respect to topological spaces based respectively on Lewis's spheres and (Stalnaker's inspired) selection functions. The dualities we show are enrichments of Stone duality between Boolean algebras with homomorphisms and Stone spaces with continuous maps, where the operator $\Box\!\!\rightarrow$ is interpreted first by means of a selection function, and then by a map associating a set of nested spheres to each element of the space. The formal work developed for the dualities also allows us to demonstrate the strong completeness of sphere models with respect to Lewis's logics. Finally, thanks to the duality results, we also clarify the role of the *limit assumption*, a condition on sphere models that has been extensively discussed in the literature (see for example [6, 2]). In particular, we will see that both the strong and weak calculi are strongly complete with respect to models that do satisfy the limit assumption; in this sense, models without the limit assumption are not really "seen" by Lewis's logics.

In conclusion, this contribution is meant to provide a logico-algebraic treatment of Lewis variably strict conditional logics. Our results aim at clarifying several ambiguities in the literature surrounding these logics, explicitly defining and refining their properties and theorems, and introducing a novel general algebraic and topological framework for their technical analysis.

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On the implicative subreducts of subresiduated lattices

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Abstract

It is known that sub-Hilbert algebras are the implicative subreducts of subresiduated lattices. In this work we give a new proof of this property by using ideas employed to represent weak Heyting algebras. We also study the lattice of relative congruences of sub-Hilbert algebras and we give a quasi-equational description of the quasivariety of sub-Hilbert algebras generated by the class of its totally ordered members.

Subresiduated lattices were introduced by Epstein and Horn [8] with the aim to study certain propositional logics defined in a language without classical implication but with a connective of implication which is called strict implication. The logics studied in [8] are examples of subintuitionistic logics, i.e., logics in the language of intuitionistic logic that are defined semantically by using Kripke models, in the same way as intuitionistic logic is defined, but without requiring of the models some of the properties required in the intuitionistic case [5, 6].

A subresiduated lattice (sr-lattice for short) [6, 8] is a pair (A, D), where A is a bounded distributive lattice, D is a bounded sublattice of A and for every $a, b \in A$ there exists the maximum of the set $\{d \in D : a \wedge d \leq b\}$, which is denoted by $a \rightarrow b$. This pair can be regarded as an algebra $(A, \wedge, \vee, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ where $D = \{a \in A : 1 \rightarrow a = a\}$. The class of sr-lattices properly contains the variety of Heyting algebras. It follows from [8, Theorem 1] that the class of sr-lattices forms a variety. A different equational base for this variety was given in [6], where this variety is presented as a subvariety of the variety of weak Heyting algebras.

Recall that S4-algebras are Boolean algebras with a unary operator $\Box$ in the language that satisfies the identities $\Box 1 = 1$, $\Box(x \wedge y) = \Box x \wedge \Box y$, $\Box x \leq x$ and $\Box x \leq \Box(\Box x)$. We say that an algebra $(A, \wedge, \vee, \rightarrow, \neg, 0, 1)$ is a Boolean subresiduated lattice (Boolean sr-lattice for short) if $(A, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra and $(A, \wedge, \vee, \rightarrow, 0, 1)$ is a sr-lattice. If $(A, \wedge, \vee, \rightarrow, \neg, 0, 1)$ is a Boolean sr-lattice then $(A, \wedge, \vee, \neg, \Box, 0, 1)$ is a $S4$-algebra, where $\Box x := 1 \rightarrow x$. Conversely, if $(A, \wedge, \vee, \neg, \Box, 0, 1)$ is a S4-algebra then $(A, \wedge, \vee, \rightarrow, \neg, 0, 1)$ is a Boolean sr-lattice, where $x \rightarrow y := \Box(\neg x \vee y)$. Moreover, the variety of boolean sr-lattices is term equivalent to the variety of S4-algebras. It can be also showed that the variety of sr-lattices coincides with the class of $\{\wedge, \vee, \rightarrow, 0, 1\}$-subreducts of Boolean sr-lattices.

A sub-Hilbert algebra [3] is an algebra $(A, \rightarrow, 1)$ of type (2,0) which satisfies the following quasi-equations:

- $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
- $x \rightarrow x = 1$,
- $x \rightarrow 1 = 1$,
- if $x \rightarrow y = 1$ and $y \rightarrow x = 1$, then $x = y$,

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- $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$.

We write sHA to indicate the class of sub-Hilbert algebras, which properly contains the variety of Hilbert algebras. The class sHA is a quasivariety which is not a variety [4]. Moreover, in sub-Hilbert algebras the binary relation $\leq$, defined by $a \leq b$ if and only $a \to b = 1$, is a partial order and 1 is the last element with respect to this order [4]. In [4] it was also proved that sub-Hilbert algebras are the implicative subreducts of sr-lattices, property which generalizes the fact that Hilbert algebras are the implicative subreducts of Heyting algebras [1, 7]. This result is proved by using properties of implicative filters of sub-Hilbert algebras, where an implicative filter of a sub-Hilbert A is a subset F of A such that $1 \in F$ and $b \in F$ whenever $a, a \to b \in F$.

In the present work we prove, following an alternative path to that given in [4] and motivated by certain constructions developed in [3, 6] for some classes of algebras, that sub-Hilbert algebras are the implicative subreducts of sr-lattices. More precisely, given $A \in$ sHA we show the following two facts: 1) it is possible to define a binary relation R on the set $\mathrm{IF}(A)$ of implicative filters of A which induces a binary operation $\Rightarrow_R$ on the set $\mathrm{IF}(A)^+$ of upsets of $(\mathrm{IF}(A), \subseteq)$ such that $(\mathrm{IF}(A)^+, \cap, \cup, \Rightarrow_R, \emptyset, \mathrm{IF}(A))$ is a sr-lattice; 2) there exists a monomorphism from A to $(\mathrm{IF}(A)^+, \Rightarrow_R, \mathrm{IF}(A))$. We also show that for every $A \in$ sHA, the lattice of relative congruences of A is order isomorphic to the lattice of open implicative filters of A, where an implicative filter F of A is said to be open if $1 \to a \in F$ whenever $a \in F$. Moreover, we study properties of the irreducible open implicative filters [1]. Finally, motivated by some results given in [2, 9], we apply properties of irreducible open implicative filters in order to give a quasi-equational description of the quasivariety of sHA generated by the class of its totally ordered members.

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[1] Let $A \in$ sHA. A proper open implicative filter P of A is called irreducible when for all F_1 and F_2 open implicative filters, if $P = F_1 \cap F_2$, then $P = F_1$ or $P = F_2$.

Finitely Weighted Kleene Algebra With Tests

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Kleene algebras, going back to [2], are algebraic structures central to automata theory, semantics of programs, and theoretical computer science in general. Kozen [4] has shown that the equational theory of Kleene algebras is complete with respect to the model based on regular languages. Kozen [5] introduces Kleene algebras with tests, a combination of Kleene algebras (programs) and Boolean algebras (tests), and shows that they have non-trivial applications in verification of imperative programs. Ésik and Kuich [3] generalize Kozen's completeness result for Kleene algebras to the case of weighted regular languages, or formal power series. In particular, their result applies to a weighted generalization of Kleene algebras where the semiring of weights is finite, commutative, zero-bounded (or positive) and partially ordered.

We establish two completeness results for a weighted generalization of Kleene algebras with tests. First, we establish completeness with respect to the algebra of weighted guarded languages using a reduction to weighted regular languages similar to the one used by Kozen and Smith [6] in their completeness proof for (non-weighted) Kleene algebras with tests. Second, we establish completeness with respect to weighted transition systems by using a Cayley-like construction going back to Pratt's work [7] on (non-weighted) dynamic algebras. In addition to the assumptions of Ésik and Kuich, however, we need to assume that the semiring of weights is also integral. These results are interesting also because of the connection between weighted Kleene algebras with tests and weighted programs [1], noted in our earlier work [8]. We also argue that finitely weighted Kleene algebras with tests are a natural framework for equational reasoning about weighted programs in cases where an upper bound on admissible weights is assumed.

A *Kleene algebra* [4] is an idempotent semiring X with a unary operation * satisfying, for all $x, y, z \in X$ the following *unrolling* (left column) and *fixpoint* laws (right column):

$$1 + (x \cdot x^*) = x^* \qquad\qquad y + (x \cdot z) \leq z \implies x^* \cdot y \leq z \tag{1}$$

$$1 + (x^* \cdot x) = x^* \qquad\qquad y + (z \cdot x) \leq z \implies y \cdot x^* \leq z\,. \tag{2}$$

a Kleene algebra with tests [5] is a Kleene algebra X with a distinguished $B \subseteq X$ such that $\langle B, +, \cdot, 0, 1\rangle$ is a subalgebra of X and a bounded distributive lattice, and $^-$ is an unary operation on B such that $x \cdot \bar{x} = 0$ and $x + \bar{x} = 1$ for all $x \in B$. Hence, B forms a Boolean algebra. Intuitively, elements of B represent *Boolean tests.* ("If b then x else y" can be expressed as $bx + \bar{b}y$ and "While b do x" as $(bx)^*\bar{b}$; partial correctness is expressed by $bx\bar{c} = 0$.)

Definition 1. *Let S be a finite semiring. A* Kleene S-algebra with tests *is a Kleene algebra with tests X together with a binary operation $\odot : X \times S \to X$ such that (the additive monoid reduct of) X forms a right S-semimodule and*

$$(xy) \odot s = x(y \odot s) = (x \odot s)y \qquad\qquad 1 \odot s^* \leq (1 \odot s)^*$$

Similar to Kleene algebras with tests, the algebraic language for Kleene S-algebras with tests is two-sorted, consisting of *tests* and *expressions*:

$$b, c := \mathtt{p} \mid \bar{b} \mid b + c \mid b \cdot c \mid \mathtt{0} \mid \mathtt{1} \qquad\qquad e, f := \mathtt{a} \mid b \mid e \odot s \mid e + f \mid e \cdot f \mid e^*$$

where $\mathtt{p} \in \Phi$ (a finite set of proposition letters), $\mathtt{a} \in \Sigma$ (a finite set of program letters) and $s \in S$. Expression $e \odot s$ means "execute e and add s to the weight of the current computation".

An atom over Φ is a finite sequence of literals over Φ containing exactly one of $\mathtt{p}$ and $\bar{\mathtt{p}}$ for each $\mathtt{p} \in \Phi$. A guarded string is a string of the form $G_1\mathtt{a}_1G_2 \ldots \mathtt{a}_{n-1}G_n$ where the G's are atoms and the $\mathtt{a}$'s are program letters. Fusion product $wG \diamond Hu$ of guarded strings is undefined if $G \neq H$ and wGu otherwise. The set of *guarded formal power series* over a finite semiring S is the set of mappings from the set of guarded strings to S. The *rational operations* on guarded f.p.s. are defined point-wise as follows:

$$(r_1 + r_2)(w) = r_1(w) + r_2(w) \qquad (r_1 \cdot r_2)(w) = \sum \{r_1(v_1) \cdot r_2(w_2) \mid w = v_1 \diamond v_2\}$$

$$(r \odot s)(w) = r(w) \cdot s \qquad r^*(w) = \sum_{n \in \omega} r^n(w)$$

where $r^0 = 1$ and $r^{n+1} = r^n \cdot r$. (Note that the sum is defined since S is assumed to be finite.) A *polynomial* is any guarded f.p.s. r such that the set of guarded strings w where $r(w) \neq 0$ is finite. The set of *rational guarded f.p.s.* is the least set of guarded f.p.s. that contains all polynomials and is closed under the rational operations. The set of rational guarded f.p.s. over S forms a Kleene S-algebra with tests.

Theorem 1. *The equational theory of Kleene S-algebras with tests coincides with the equational theory of the algebra of rational guarded f.p.s. over S.*

An S-transition system is a set with a collection of S-weighted binary relations $M(\mathtt{a})$ on the set for $\mathtt{a} \in \Sigma$ and $\{0, 1\}$-weighted diagonal relations $M(\mathtt{p})$ for $\mathtt{p} \in \Phi$. Binary relations $M(e)$ for arbitrary expressions are defined as expected using familiar matrix operations.

Theorem 2. *An equation $e \approx f$ is valid in all Kleene S-algebras with tests iff $M(e) = M(f)$ in all S-transition systems.*

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The Priestley duality for $\prec$-distributive $\vee$-predomains

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In his celebrated paper [13], M. Stone gave a topological representation for Boolean algebras, linking the worlds of topology and lattices together for the first time. He showed that the category of Boolean algebras and lattice homomorphisms is dually equivalent to that of Stone spaces and continuous maps. Shortly afterwards, Stone [14] extended this topology-lattice duality for the larger class of bounded distributive lattices by introducing the nowadays known spectral spaces. With the aim of connecting lattices to the classical Hausdorff topological spaces, Priestley [11] used ordered topological spaces and proposed another topological representation, nowadays known as Priestley spaces, for bounded distributive lattices. She established a dual equivalence between the category of bounded distributive lattices with lattice homomorphisms and that of Priestley spaces with monotone continuous maps.

Duality theory between topology and lattices has been extensively applied in many fields such as topology, functional analysis and logic, among others, and the study of topological representations for general partially ordered structures has been attracting wide attention. Starting from the 1970s, Stone duality has been generalized for spatial frames [9, 10], semilattices [5] and other classes of posets [6, 8]. Indeed, Gräzter [6] removed binary infima from bounded distributive lattices and obtained a topological representation for bounded distributive join-semilattices. Hofmann and Lawson [8] developed a Stone duality for continuous frames. Moreover, scholars have extended Priestley duality to bounded lattices [15] and to other classes of posets [1, 2, 7]. In particular, Hansoul and Poussart [7] proposed a Priestley-type topological representation for bounded distributive sup-semilattices. Besides, there are some works obtained by restricting Priestley duality to subcategories of bounded distributive lattices and lattice homomorphisms. Both the Pultr-Sichler duality [12] for frames and the Bezhanishvili-Melzer duality [3] for continuous frames fall into this category, for instance.

Recently, Bice [4] unified distributive join-semilattices and continuous frames as $\prec$-distributive $\vee$-predomains and further developed a Stone duality for $\prec$-distributive $\vee$-predomains, which is a common extension of the Hofmann-Lawson duality and the Grätzer duality. Precisely, he proved that the category of locally compact sober spaces with a base closed for finite unions is equivalent to that of $\prec$-distributive $\vee$-predomains. However, a Priestley duality for $\prec$-distributive $\vee$-predomains is unknown, and Bice left it open in [4].

In this talk, we give an affirmative answer to the above question of Bice. We introduce DP-compact pospaces as follows.

Definition 1. We call a tuple $(X, \tau, \leqslant, X_1, \beta)$ *DP-compact pospace* if

(1) $(X, \tau, \leqslant)$ is a compact pospace,

(2) X_1 is dense and order generating,

(3) β is composed by admissible lower open sets and closed under finite unions,

(4) $x \in X_1$ if and only if $\{U \in \beta \mid x \in U\}$ is a neighborhood base of x with respect to (X, τ^b).

Here, a lower open set U is *admissible* if $U = \downarrow(U \cap X_1)$. τ^b is the topology consisting of all lower open sets in $(X, \tau, \leqslant)$.

Then we establish a one-to-one correspondence between $\prec$-distributive $\vee$-predomains and DP-compact pospaces. To develop a dual equivalence, we further propose DP-morphisms and $\prec$-morphisms. Let **DPCP** be the category of DP-compact pospaces with DP-morphisms and **DP** be the category of $\prec$-distributive $\vee$-predomains with $\prec$-morphisms. Then we obtain our main theorem:

Theorem 2 (Main theorem)**.** *Categories* **DP** *and* **DPCP** *are dually equivalent.*

In addition, we would like to stress that our results restrict to the Hansoul-Poussart duality [7] and a Priestley duality for continuous frames. The Priestley-type topological representations of continuous frames are CF-compact pospaces defined as follows.

Definition 3. We call a tuple $(X, \tau, \leqslant, X_1)$ a *CF-compact pospace* if

(1) $(X, \tau, \leqslant)$ is a compact pospace,

(2) X_1 is dense and order generating,

(3) $x \in X_1$ if and only if $\{U \in \tau^b \mid x \in U = \downarrow(U \cap X_1)\}$ is a neighborhood base of x with respect to (X, τ^b).

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A Logic of Belief Revision in Simplicial Complexes

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In this paper, we present a model of epistemic modal logic in simplicial semantics with the aim of motivating a new interpretation of belief revision by way of imaging. Our model modifies and expands on recent papers by Eric Goubault, Doman Kniazev, Sergio Rajsbaum, Jérémy Ledent, and Hans van Ditmarsch, which use simplicial complexes as a semantics for epistemic modal logic, leveraging their special structure not present in the usual Kripke setting.[3][2][1][6] Our motivation is similar: we believe simplicial semantics affords us tools for analyzing belief revision not obviously present in Kripke models.

In simplicial semantics, possible worlds are not treated as primitive. Instead, they can be identified with sets of agent perspectives. Perspectives are given, for technical reasons, as sets of literals. In our model, we construct such "worlds" by taking consistent sets of perspectives, with one perspective uniquely associated with each agent. We will call these "worlds" facets. Given such a facet in a simplicial complex S, call it X, we say that Y is *accessible* for agent a from X if X and Y share their unique a-perspective. As is standard, one says that a formula $K_a\varphi$ is true if and only if φ is true at all facets Y which are accessible for agent a from X. One can show, via a categorical equivalence, that this framework is equivalent to a particular class of Kripke models where the accessibility relations form partitions (i.e., models for **S5**).

This is similar to what has been done in the previous literature. There, the simplicial complexes are used to model knowledge. [2][3][1][4][5][6][7][9] If we are to use simplicial complexes as a model for belief, the fact that the axiom T is sound is undesirable. It is easy to see why it is sound in the usual setup, as X is always a-accessible from X.

To get around this, we will introduce distinct simplicial complexes for each agent. More specifically, given a set of perspectives, each agent will have their own simplicial complex over this set, call it S_a. All of the S_a will be subcomplexes of a background complex, call it S, which is not specific to any agent. We then modify the definition of a-accessible facets as follows. Given facets X and Y in S, we say that Y is *accessible** for agent a from X if and only if Y is a facet in S_a and X and Y share their unique a-perspective. If we say that $B_a\varphi$ is true at X if and only if φ is true at all Y that are *a-accessible** from X, It's easy to show that this makes sound **K45**. Additionally, it's easy to see why the axiom T is not sound, as if X is not a facet in S_a, then X is not *a-accessible** from X.

Motivated by the idea of charity towards other agents, we argue that a good notion of "nearness" between two worlds, in the sense of Lewis' work on imaging, in the simplicial setting is given by the size of their intersection.[10] That is, worlds which share more perspectives are closer. More specifically, suppose the formula φ is publicly announced. Then we can define an imaging function R which replaces every facet X in S_a with a set of facets $R(X)$, which consists of those facets which 1: satisfy φ, 2: share their unique a-perspective, and 3: are such that for any facet $Y \in R(X)$, and any facet Z satisfying conditions 1 and 2, $|X \cap Y| \geq |X \cap Z|$. We explore variations of this imaging function and some of their consequences. For instance, we could restrict $R(X)$ to always be a set of facets from the background complex S, and furthermore have S eliminate facets which contradict announced information with each announcement. In this way, S acts as a kind of "memory" for the agents. Another option would be to say that if there are facets in S_a which share the a perspective with X and satisfy φ, then $R(X)$ should be this set, regardless of the size of the intersection. We show, specifically, that a variant of this second update mechanism is a "nested sphere" model in the sense of Grove, 1988, and therefore satisfies the AGM axioms. [8]

Furthermore, we give soundness proofs for the knowledge and belief modalities in our models. We interpret the belief modality using *accessibility**, and the knowledge modality using *accessibility* for the background complex S. Specifically, for the following language:

$$\varphi ::= P|\bot|\varphi \to \varphi|B_a\varphi|K_a\varphi$$

our simplicial semantics is sound with respect to the axioms of propositional logic, **S5** for the K_a modality, **K45** for the B_a modality, the axiom $K_a\varphi \to B_a\varphi$, and the following axiom for each modality, and any atomic formula P, which we call **NU** for "No Uncertainties":

$$\mathbf{NU} : P \to \bigvee_{a \in A} K_a P$$

Axioms similar to **NU** appear throughout the simplicial literature. [1].

One easy way to see that these soundness results hold is to proceed as much of the existing literature does, by demonstrating a logic preserving categorical equivalence between a category whose objects are our simplicial models, and a category whose objects are a class of Kripke models, where these Kripke models make sound these same axioms. We conclude by discussing how one can extend this equivalence to a proof of completeness.

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Weil algebras and varieties of rigs

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As explained in [3], some of Grothendieck's algebro-geometric constructions may be abstracted to the context of extensive categories. A category C with finite coproducts is extensive if the canonical functor $\mathsf{C}/X \times \mathsf{C}/Y \to \mathsf{C}/(X+Y)$ is an equivalence for every pair of objects X, Y in C. Extensivity attempts to make explicit a most basic property of (finite) coproducts in categories 'of spaces'. For instance, the category of topological spaces and continuous functions between them is extensive; the category of groups is not. It easily follows that if C is extensive then for any $X \in \mathsf{C}$ the category X/C is extensive [1].

Experience indeed confirms that conceiving an extensive category as a category 'of spaces' is a useful conceptual guide. Essential to the development of Algebraic Geometry is the fact that the opposite of the category of (commutative unital) rings, is extensive.

A category C is coextensive when its opposite category C^{op} is coextensive. In this work we examine the variety of algebras known as *rigs*, denoted Rig, which are commutative semirings with (additive and multiplicative) unit. Of particular interest are those subvarieties $2\mathsf{Rig}$ of (additively) idempotent rigs, as well as the variety iRig of *integral rigs*; those satisfying $1 + x \approx 1$. Such classes play an important role, for instance, in non-classical logics in that these algebras are exactly the (integral) join-semilattice reducts of (pointed) commutative residuated lattices, or $\mathsf{FL_e}$-algebras (respectively, $\mathsf{FL_{ew}}$), semantics for certain extensions of the Full Lambek calculus. Viewed as categories, these classes are coextensive (see [2, 4]), and thus admit to the prospect of geometric content.

Let C be a category with a terminal object 1. If X is an object of C, a *point* of X is an arrow $1 \to X$. An object is called *Weil* if it has a unique arrow to the terminal object. At least in the case when the category is a variety of algebras, the terminal object is the free 0-generated algebra. In the case of the variety of rigs, the terminal object is the rig of natural numbers $\mathbb{N}$, while for (non-trivial) subvarieties of 2-rigs the terminal object is always the two element chain $\mathsf{2}$. We note that there is no finite Weil algebra in the in Rig.

An arrow $f\colon X \to Y$ in C is called *constant* if it factors through 1. More generally, an arrow $f\colon X \to Y$ is called a *pseudo-constant* if it coequalizes all the points of X. That is,

$$1 \underset{a}{\overset{b}{\rightrightarrows}} X \xrightarrow{\ f\ } D$$

for every pair of points $a, b\colon 1 \to X$, one has $f(a) = f(b)$. Of course, every constant is a pseudo-constant.

Let us write Aff for the opposite of $2\mathsf{Rig}$, and if A is an object in $2\mathsf{Rig}$, let us write A' for the corresponding object in Aff. Trivially, points of A' in Aff are in bijective correspondence with maps $A \to \mathsf{2}$ in $2\mathsf{Rig}$. So, for example, A is a Weil 2-rig iff A' has exactly one point. A map $f\colon A \to B$ is called *pseudo-stant* if for every $g, h\colon B \to \mathsf{2}$ one has $g \circ f = h \circ f$. So, a map is pseudo-stant in the category iR if and only if the corresponding $B' \to A'$ in Aff is a pseudo-constant. Experience with Set suggests that pseudo-constants are constant, but this is too naive. What is sometimes the case in categories of spaces is that the image of a pseudo-constant has exactly one point. This is the content of the following question.

Question 1. Let $\mathcal{V}$ be a variety of rigs. Let $f\colon A \to B$ be such that for every $g, h\colon B \to \mathsf{2}$, $g \circ f = h \circ f\colon A \to \mathsf{2}$. Is it the case that f factors through one Weil algebra in $\mathcal{V}$?

Part of this work is devoted to providing an answer to the above question. In the case for classes of 2-rigs, in particular irigs, we answer this question in the affirmative. This is, in part, a consequence of the following characterization.

Theorem 2. *Let R be any 2-rig. Then the following are equivalent.*

1. *R is a Weil 2-rig.*
2. *R has a unique prime ideal closed under $\leq$.*
3. *R satisfies the following:*

$$\text{For all } x \in R, \exists n \in \mathbb{N}, x^n \leq 0 \quad \text{or} \quad \exists r \in R, 1 \leq rx.$$

where $\leq$ is the partial order defined via $x \leq y$ iff $x + y = y$ in R.

Moreover, the theorem above can be used to establish that the variety of 2-rigs is generated by a its Weil members, in particular this class can be taken to consist of finite algebras of a certain form.

Theorem 3. *For $\mathcal{V}$ taken to be the variety of 2-rigs or the variety of integral rigs, $\mathcal{V}$ is generated by a class of its finite Weil members. Specifically, each finitely generated free-algebra is a subdirect product of finite Weil algebras in $\mathcal{V}$ [satisfying a stronger version of item (3)].*

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The $(\infty, 2)$-category theory of internal $(\infty, 1)$-categories

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Internal category theory is the study of the 2-category $\mathbf{Cat}(\mathcal{C})$ of internal categories, internal functors and internal natural transformations respective a base category $\mathcal{C}$. The Yoneda embedding $y\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ can be understood as an externalization functor of finite limit structures in $\mathcal{C}$ (made precise in [5]); it can be shown to induce an embedding

$$y_*\colon \mathbf{Cat}(\mathcal{C}) \to \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Cat})$$

of 2-categories, at times also referred to as the externalization functor of the category $\mathcal{C}$ (see e.g. [2]). On the flipside, internalization is the practice of reflecting properties and indexed structures along y_* whenever possible.

Given an ∞-categorical base $\mathcal{C}$ (usually an ∞-topos or at least left exact), the ∞-category $\mathrm{Cat}_\infty(\mathcal{C})$ of $\mathcal{C}$-internal ∞-categories is well-known, and internal constructions of internal ∞-categories are pervasive in the higher categorical literature. A systematic study of internal ∞-category theory as such however is not; indeed a definition of an according ∞-categorical enrichment $\mathbf{Cat}_\infty(\mathcal{C})$ by hand is much less tangible than in the ordinary case. Thus, following [6], in this talk we instead discuss the ∞-categorical externalization functor

$$y_*\colon \mathrm{Cat}_\infty(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty)$$

first, and use it to *define* the $(\infty, 2)$-category $\mathbf{Cat}_\infty(\mathcal{C})$ of $\mathcal{C}$-internal ∞-categories as embedded in the $(\infty, 2)$-category $\mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Cat}_\infty)$ of $\mathcal{C}$-indexed ∞-categories. We show various formal $(\infty, 2)$-categorical closure properties of $\mathbf{Cat}_\infty(\mathcal{C})$ under the assumption of various suitable $(\infty, 1)$-categorical closure properties of $\mathcal{C}$. The main theorem states that the $(\infty, 2)$-category $\mathbf{Cat}_\infty(\mathcal{C})$ is a full sub-∞-cosmos of $\mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Cat}_\infty)$ which is closed under all limits (and exponentials) whenever $\mathcal{C}$ is complete (and cartesian closed). It thus defines a (cartesian closed) ∞-cosmos in the sense of [3]. This means that a plethora of indexed ∞-categorical constructions defined over a collection of internal ∞-categories indexed over such a base $\mathcal{C}$ can be internalized in $\mathcal{C}$ automatically. We furthermore characterize the objects of $\mathbf{Cat}_\infty(\mathcal{C})$ by means of a Yoneda lemma that expresses indexed diagrams of internal shape over $\mathcal{C}$ in terms of an ∞- categorical totalization, and discuss applications.

Lastly, we relate the general theory developed to this point to results in the model categorical literature. We show that every model category $\mathbb{M}$ gives rise to a "hands-on" ∞-cosmos $\mathbf{Cat}_\infty(\mathbb{M})$ (of not-necessarily cofibrant objects) directly by restriction of the Reedy model structure on $\mathbb{M}^{\Delta^{op}}$. We then define an according right derived model categorical externalization functor, and use it to show that the ∞-categorical and the model categorical constructions correspond to one another whenever $\mathcal{C}$ is presentable and $\mathbb{M}$ is a suitable presentation thereof. This shows that the theory presented in this talk recovers as special cases various well-known constructions in the model categorical literature. This for instance includes Dugger's simplicial replacements of model categories [1], Toën's framework for theories of $(\infty, 1)$-categories [7], as well as Riehl and Verity's ∞-cosmoses of Rezk-objects [4] among others.

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Raney extensions of frames as pointfree T_0 spaces

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Raney duality, as illustrated in [3], consists of a dual equivalence of categories between Raney algebras and T_0 spaces. The dual equivalence sends a space X to the embedding $\Omega(X) \subseteq \mathcal{U}(X)$, where $\Omega(X)$ is its lattice of opens and $\mathcal{U}(X)$ its lattice of saturated sets (the upper sets in the specialization order). In this setting, all Raney algebras are, so to speak, spatial: there are no Raney algebras which are not $\Omega(X) \subseteq \mathcal{U}(X)$ for some space X. We propose to extend Raney duality by extending the category of Raney algebras to a more pointfree category. We consider *Raney extensions*, as introduced in [7], pairs (L, C) where C is a coframe, and $L \subseteq C$ is a frame that meet-generates C such that the inclusion preserves the frame operations as well as the strongly exact meets. Raney extensions are the objects of the category **Raney**, whose morphisms are coframe maps which restrict to frame maps on the first components. We have the following.

Proposition 1. *There is an adjunction $\Omega_R : \mathbf{Top} \leftrightarrows \mathbf{Raney}^{op} : \mathsf{pt}_R$, where $\Omega_R(X) = (\Omega(X), \mathcal{U}(X))$ for all spaces X. In* **Top**, *the fixpoints are the T_0 spaces.*

We will see that the opposite of the frame $\mathsf{Filt}_{\mathcal{SE}}(L)$ of strongly exact filters and the opposite of the frame $\mathsf{Filt}_{\mathcal{E}}(L)$ of exact filters studied in [6] and [5] are, respectively, the largest and the smallest Raney extension of some frame L. These two frames are known to be, respectively, anti-isomorphic to the collection $\mathsf{S}_{\mathfrak{o}}(L)$ of fitted sublocales and isomorphic to the collection $\mathsf{S}_{\mathfrak{c}}(L)$ of joins of closed sublocales. In [7] the following is shown.

Theorem 2. *For a frame L, the largest Raney extension on it is $(L, \mathsf{S}_{\mathfrak{o}}(L))$, and the smallest one is $(L, \mathsf{S}_{\mathfrak{c}}(L)^{op})$.*

A topological space X is T_D if for all $x \in X$ there are opens $U, V \subseteq X$ such that $\{x\} = U \setminus V$. This axiom is introduced in [1]. In [2], it is shown that the axiom is in a certain sense dual to sobriety, in fact the following is shown.

- A space X is sober if and only if whenever a subspace inclusion $X \subseteq Y$ induces a frame isomorphism $\Omega(X) \cong \Omega(Y)$, then that inclusion is the identity.
- A space X is T_D if and only if whenever a subspace inclusion $Y \subseteq X$ induces a frame isomorphism $\Omega(Y) \cong \Omega(X)$, then that inclusion is the identity.

In [2], the definition of the T_D spectrum $\mathsf{pt}_D(L)$ of a frame L is given. With the following result, we find another sense in which sobriety and the T_D property are dual of one another.

Proposition 3. *For a frame L, the spectrum of the smallest Raney extension $(L, \mathsf{S}_{\mathfrak{c}}(L)^{op})$ is its T_D spectrum $\mathsf{pt}_D(L)$. The spectrum of the largest one $(L, \mathsf{S}_{\mathfrak{o}}(L))$ is the classical spectrum $\mathsf{pt}(L)$. Furthermore, for any Raney extension (L, C) we have subspace inclusions $\mathsf{pt}_D(L) \subseteq \mathsf{pt}_R(L, C) \subseteq \mathsf{pt}(L)$, up to isomorphism.*

In the context of Raney extensions, unlike that of frames, we have a natural version of the T_1 axiom. Because a space X is T_1 if and only if all subsets are saturated, that is, $\mathcal{U}(X) = \mathcal{P}(X)$, we define a Raney extension (L, C) to be T_1 if C is a Boolean algebra. This enables us to find a characterization of subfitness for frames as the weakest possible version of the T_1 axiom.

Theorem 4. *For a frame L, the following are equivalent.*

- *The frame L is subfit.*
- *The frame L admits a T_1 Raney extension.*
- *The frame L admits a unique T_1 Raney extension.*
- *The Raney extension $(L, \mathsf{S_c}(L))$ is T_1.*

We will see that Raney extensions generalize canonical extensions for distributive lattices, and for locally compact frames (see [4]). We say that an element $c \in C$ for a Raney extension (L, C) is *compact* if it is inaccessible by directed joins of families in L. We say that a Raney extension is *algebraic* when it is generated by its compact elements. We have the following ([7]).

Proposition 5. *For a pre-spatial frame L, the canonical extension (L, L^δ) is the free algebraic Raney extension over it.*

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Decompositions of locally integral involutive residuated structures

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An *involutive partially-ordered semigroup* (*ipo-semigroup*) is a structure of the form $\mathbf{A} = (A, \leq, \cdot, \sim, -)$ such that $(A, \leq)$ is a partially ordered set and $(A, \cdot)$ is a semigroup with two order-reversing operations $\sim$ and $-$ satisfying *involution* $\sim\!-x = x = -\!\sim\! x$ and *rotation* $x \cdot y \leq z \iff y \cdot \sim z \leq \sim x \iff -z \cdot x \leq -y$. In the case that the semigroup has an identity, we call it an *ipo-monoid*. An ipo-semigroup in which the partial order is a lattice order is called an *iℓ-semigroup*.

In the presence of order-reversal and involution, rotation is equivalent to *residuation*:

$$xy \leq z \iff x \leq -(y \cdot \sim z) \iff y \leq \sim(-z \cdot x).$$

Thus, the multiplication of every ipo-semigroup is residuated in both arguments, with *left* and *right residuals* given by $z/y = -(y \cdot \sim z)$ and $x\backslash z = \sim(-z \cdot x)$, respectively.

We say that an ipo-monoid is *integral* if the global identity 1 is also the top element. In this case, $x\backslash x = 1 = x/x$. More generally, an ipo-semigroup $\mathbf{A}$ has local identities if $x\backslash x = x/x$ for all x, in which case we denote this element by 1_x, and $1_x \cdot x = x$. If, moreover, elements are bounded by their local identities ($x \leq 1_x$), the local identities are positive ($y \leq 1_x \cdot y$), and $x\backslash 1_x = 1_x$, then we say that $\mathbf{A}$ is *locally integral*.

We show that every locally integral ipo-semigroup $\mathbf{A}$ decomposes uniquely into a Płonka sum over a semilattice directed system of integral ipo-monoids. We also solve the reverse problem, that is, we provide necessary and sufficient conditions so that the glueing of a system of integral ipo-monoids becomes an ipo-semigroup. This is a generalization of the results in [1], in which the decomposition and glueing results are proven for locally integral ipo-monoids.

Commutative idempotent locally integral ipo-semigroups are called *locally integral ipo-semilattices* and decompose into a system of Boolean algebras. A structural description of finite commutative idempotent involutive residuated lattices (unital iℓ-semilattices) is given in [2]. We also describe a dual representation for a class containing all finite locally integral ipo-semilattices via semilattice directed systems of partial functions between sets.

This is joint work with José Gil-Férez and Peter Jipsen.

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A Categorical Characterization of the Low-Complexity Functions

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A function is primitive recursive iff it is representable by a map on the parameterized initial $F_{\mathbf{N}}$-algebra of any cartesian category (if it exists), where $F_{\mathbf{N}}(X) = 1+X$. In this talk, following the philosophy of predicativism, we weaken the definition of a parameterized initial F-algebra to introduce a new notion called a *predicative F-scheme*, for any endofunctor F. Then, we show that the predicative $F_{\mathbf{N}}$-scheme (resp. predicative $F_{\mathbf{W}}$-scheme, where $F_{\mathbf{W}}(X) = 1 + X + X$) naturally captures the class of all linear space (resp. polynomial time) computable functions as its all and only representable functions. In the rest of this extended abstract, we will present the definitions of predicative F-schemes and representability to make the above points more formal.

First, we need to recall some basic definitions. Let $\mathcal{C}$ be a cartesian category (i.e., with all finite products), $F : \mathcal{C} \to \mathcal{C}$ be a functor and X be an object in $\mathcal{C}$. By an *F-algebra in $\mathcal{C}$ with parameters in X*, we mean the tuple $\mathbf{A} = (X, A, a)$, where $a : X \times F(A) \to A$ is a map in $\mathcal{C}$. The object A is called the *carrier* of $\mathbf{A}$ and is denoted by $|\mathbf{A}|$. When $X = 1$, an F-algebra with parameters in X is simply called an F-algebra. For any two F-algebras $\mathbf{A} = (X, A, a)$ and $\mathbf{B} = (X, B, b)$ in $\mathcal{C}$ with parameters in X, by an *F-homomorphism*, we mean a $\mathcal{C}$-map $f : A \to B$ such that the following diagram commutes:

$$\begin{tikzcd}
X \times F(A) \arrow[r, "a"] \arrow[d, "id_X \times F(f)"'] & A \arrow[d, "f"] \\
X \times F(B) \arrow[r, "b"'] & B
\end{tikzcd}$$

It is clear that F-algebras in $\mathcal{C}$ with parameters in X together with F-homomorphisms form a category denoted by $\mathbf{Alg}^F_X(\mathcal{C})$. Moreover, the assignment $|-| : \mathbf{Alg}^F_X(\mathcal{C}) \to \mathcal{C}$ mapping an F-algebra with parameters in X to its carrier and an F-homomorphism to itself is a functor. Also, note that any $g : X \to Y$ in $\mathcal{C}$ induces a canonical functor $g^* : \mathbf{Alg}^F_Y(\mathcal{C}) \to \mathbf{Alg}^F_X(\mathcal{C})$.

Definition 1. Let $\mathcal{E}$ be a cartesian category, $\mathcal{D}$ be its (not necessarily full) cartesian subcategory, $i : \mathcal{D} \to \mathcal{E}$ be the inclusion functor preserving all finite products, and $F : \mathcal{E} \to \mathcal{E}$ be a functor whose restriction to $\mathcal{D}$ lands in $\mathcal{D}$ itself. An object I in $\mathcal{E}$ is called the *F-scheme of $\mathcal{D}$ in $\mathcal{E}$*, if for any $X \in \mathcal{D}$, the object $X \times I$ is the limit of the diagram $i|-| : \mathbf{Alg}^F_X(\mathcal{D}) \to \mathcal{E}$ via the cone $\langle r_{X,\mathbf{A}} \rangle_{\mathbf{A} \in \mathbf{Alg}^F_X(\mathcal{D})}$ and for any $\mathcal{D}$-map $f : X \to Y$ and any F-algebra $\mathbf{A}$ in $\mathcal{D}$ with parameters in Y, the following diagram commutes:

$$\begin{tikzcd}
I \times X \arrow[d, "id_I \times f"'] \arrow[dr, "r_{X, f^*\mathbf{A}}"] & \\
I \times Y \arrow[r, "r_{Y,\mathbf{A}}"'] & |\mathbf{A}|
\end{tikzcd}$$

The F-scheme of $\mathcal{D}$ in $\mathcal{E}$ is meant to formalize the common *scheme* of all F-algebras of $\mathcal{D}$ (with parameters) inside the possibly greater category $\mathcal{E}$. Using the universality of the limit, one can easily show that there is a canonical F-algebra structure $a_I : F(I) \to I$ on I, whose composition with the projection provides an F-algebra structure on $I \times X$ with parameters in X. It is also easy to see that this algebraic structure makes all $r_{X,\mathbf{A}}$'s into F-homomorphisms.

In the special case when $\mathcal{E} = \mathcal{D}$, the F-scheme of $\mathcal{D}$ in $\mathcal{D}$ is nothing but the initial F-algebra:

Theorem 2. *Let $\mathcal{D}$ be a finitely complete category. If I is the F-scheme of $\mathcal{D}$ in $\mathcal{D}$, then the F-algebra $a_I : F(I) \to I$ is the parameterized initial F-algebra in $\mathcal{D}$. Conversely, if $\mathbf{A}$ is the parameterized initial F-algebra in $\mathcal{D}$, then the object $|\mathbf{A}|$ together with its unique F-homomorphisms into the F-algebras of $\mathcal{D}$ (with parameters) is the F-scheme of $\mathcal{D}$ in $\mathcal{D}$.*

In the general situation when $\mathcal{E}$ is different from $\mathcal{D}$, we need to add an additional property, called the *approximability*, to gain a more well-behaved F-scheme. Roughly speaking, although the limit of the diagram $i|-| : \mathbf{Alg}^F_X(\mathcal{D}) \to \mathcal{E}$ may not belong to $\mathcal{D}$, we want it to be approximable by the objects inside the smaller category $\mathcal{D}$. More formally, the category $Alg^F_X(\mathcal{D})$ is called approximable iff there is a directed family $\{\mathcal{S}_j\}_{j \in J}$ of classes of morphisms of $\mathcal{D}$ (not necessarily closed under composition) such that it covers the whole $Morph(\mathcal{D})$ and the restriction of $Alg^F_X(\mathcal{D})$ to $\mathcal{S}_j$ has an initial element, for any $j \in J$. Unfortunately, the fully formal definition of approximability is beyond the scope of this short abstract. The reason is some subtleties in the definitions of the restriction, the initial element and the compatibility in the parameter object X, all because the $\mathcal{S}_j$'s are not necessarily closed under the composition.

Having approximability defined, the F-scheme of $\mathcal{D}$ in $\mathcal{E}$ is called *predicative* if $Alg^F_X(\mathcal{D})$ is approximable.

Now, we turn to the representability. Let $\mathbf{N} = (\mathbb{N}, s, 0)$ and $\mathbf{W} = (\mathbb{W}, s_0, s_1, \epsilon)$ be the usual algebras of natural numbers and binary strings, where $s(n) = n + 1$, $s_0(w) = w0$, $s_1(w) = w1$ and ϵ is the empty string. In the rest, let us assume that $\mathcal{D}$ and $\mathcal{E}$ are both cartesian and cocartesian categories, $i : \mathcal{D} \to \mathcal{E}$ preserves these structures and $F_{\mathbf{N}} : \mathcal{E} \to \mathcal{E}$ and $F_{\mathbf{W}} : \mathcal{E} \to \mathcal{E}$ be the functors defined by $F_{\mathbf{N}}(X) = 1 + X$ and $F_{\mathbf{W}}(X) = 1 + X + X$. It is possible to represent any element $n \in \mathbb{N}$ (resp. $w \in \mathbb{W}$) as a map in $Hom_{\mathcal{E}}(1, I)$, if I is the $F_{\mathbf{N}}$-scheme (resp. $F_{\mathbf{W}}$-scheme) of $\mathcal{D}$ in $\mathcal{E}$. Denote this canonical representation by $\bar{n}$ (resp. $\bar{w}$). Similarly, we say that an $\mathcal{E}$-map $f : I^k \to I$ represents a function $\varphi : \mathbb{N}^k \to \mathbb{N}$ if the following commutes:

$$\begin{tikzcd}
1 \arrow[d, "\langle \bar{n}_1, \ldots, \bar{n}_k \rangle"'] \arrow[dr, "\overline{\varphi(n_1, \ldots, n_k)}"] & \\
I^k \arrow[r, "f"'] & I
\end{tikzcd}$$

for any $(n_1, \ldots, n_k) \in \mathbb{N}^k$. One can have a similar definition replacing $\mathbb{N}$ by $\mathbb{W}$. Now, we are finally ready to present our main result:

Theorem 3. (*i*) *A function $\varphi : \mathbb{N}^k \to \mathbb{N}$ is linear space computable iff it is representable as a map on the predicative $F_{\mathbf{N}}$-scheme of $\mathcal{D}$ in $\mathcal{E}$, for any $\mathcal{D}$ and $\mathcal{E}$.*

(*ii*) *A function $\varphi : \mathbb{W}^k \to \mathbb{W}$ is polynomial time computable iff it is representable as a map on the predicative $F_{\mathbf{W}}$-scheme of $\mathcal{D}$ in $\mathcal{E}$, for any $\mathcal{D}$ and $\mathcal{E}$.*

On Geometric Implications

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It is a well-known fact that although the poset of opens of a topological space is a Heyting algebra, its Heyting implication is not necessarily stable under the inverse image of continuous functions and hence is not a geometric concept. This leaves us wondering if there is any stable family of implications that can be safely called geometric. In this talk, after providing a formalization for geometricity of a family of implications over a category of spaces, we first present a classification for all geometric families over a given subcategory of **Top** satisfying some closure properties and then we show that over the full category **Top**, there is only one geometric family, consisting of trivial implications in a certain sense described below. In the rest of this extended abstract, we will present the formal version of the classification we mentioned above. Let us first start with the abstract notion of implication.

Definition 1. Let $\mathcal{A} = (A, \leq, \wedge, \vee, 1, 0)$ be a bounded distributive lattice. A binary operator $\to$ over $\mathcal{A}$, decreasing in its first argument and increasing in its second is called an *implication* over $\mathcal{A}$ if $a \to a = 1$, for any $a \in \mathcal{A}$ and $(a \to b) \wedge (b \to c) \leq a \to c$, for any $a, b, c \in \mathcal{A}$. An implication is called *weakly boolean* if $a \to b = (a \to 0) \vee b$, for any $a, b \in \mathcal{A}$. If $\to$ is an implication over the lattice of the opens of a space X, denoted by $\mathcal{O}(X)$, then the pair $(X, \to)$ is called a *strong space*. A *strong space map* is a continuous map between spaces such that its inverse image preserves the implication.

Example 2. Over any bounded distributive lattice $\mathcal{A}$, there is a *trivial implication* defined by $a \to_t b = 1$, for any $a, b \in \mathcal{A}$. The Boolean and the Heyting implications are also implications. Notice that the trivial and the boolean implications are weakly boolean.

The second element we must present is the geometricity. Intuitively, geometricity is the stability of a family of implications under the inverse image of a family of continuous functions. Therefore, to formalize this notion, we need to be precise about two ingredients: the continuous maps we use and the family of implications we choose. For the former, it is reasonable to start with a subcategory $\mathcal{S}$ of **Top** to have a relative version of geometricity. For the latter, as any implication must be over a space in this case, a natural formalization of a family of implications is some sort of fibration that to each space X in $\mathcal{S}$ assigns a fiber of strong spaces over X. Having these two ingredients fixed, the geometricity simply means the stability of the fibres under the inverse image of the maps in $\mathcal{S}$. In other words, it states that for any map $f : X \to Y$ in $\mathcal{S}$, the inverse image map f^{-1} must map a fiber over Y into a fiber over X. The following is the formalization of this idea.

Definition 3. Let $\mathcal{S}$ be a (not necessarily full) subcategory of **Top**. A category $\mathcal{C}$ of strong spaces is called *geometric over* $\mathcal{S}$, if the forgetful functor $U : \mathcal{C} \to \mathbf{Top}$ mapping $\mathcal{C}$ into $\mathcal{S}$, is surjective on the objects of $\mathcal{S}$, and for any object $(Y, \to_Y)$ in $\mathcal{C}$, any object X in $\mathcal{S}$ and any map $f : X \to Y = U(Y, \to_Y)$ in $\mathcal{S}$, there exists an object $(X, \to_X)$ in $\mathcal{C}$ such that f induces a strong space map $f : (X, \to_X) \to (Y, \to_Y)$ in $\mathcal{C}$:

$$\begin{tikzcd}
\mathcal{C} \arrow[dd, "U"'] & (X, \to_X) \arrow[r, "f"] \arrow[dd, squiggly] & (Y, \to_Y) \arrow[dd, squiggly] \\
& & \\
\mathcal{S} & X \arrow[r, "f"'] & Y = U(Y, \to_Y)
\end{tikzcd}$$

Note that using the functor U, the category $\mathcal{C}$ is nothing but a way to provide a fiber of strong spaces or equivalently a fiber of implications over any space in $\mathcal{S}$. Then, the conditions simply demand that the fibers and the maps between them are all lying over $\mathcal{S}$ and none of the fibers are empty and the last condition is the geometricity condition we discussed above.

Example 4. For any category $\mathcal{S}$ of spaces, let $\mathcal{S}_t$ be the category of strong spaces $(X, \to_t)$, where X is in $\mathcal{S}$ and $\to_t$ is the trivial implication together with all the maps of $\mathcal{S}$ as the morphisms. It is clear that $\mathcal{S}_t$ is a geometric category over $\mathcal{S}$. To have more examples, recall that a space X is called *indiscrete* if its only opens are $\varnothing$ and X and it is *locally indiscrete* if each $x \in X$ has an indiscrete neighbourhood. Now, if $\mathcal{S}$ only consists of locally indiscrete spaces, then there are three other degenerate geometric categories over $\mathcal{S}$. The first is the category $\mathcal{S}_b$ of strong spaces $(X, \to_b)$, where X is in $\mathcal{S}$ and $\to_b$ is the Boolean implication together with the maps of $\mathcal{S}$ as the morphisms. This category is well-defined, since the locally indiscreteness of X implies the Booleanness of $\mathcal{O}(X)$ and the inverse images always preserve all the Boolean operators. It is easy to see that $\mathcal{S}_b$ is actually geometric over $\mathcal{S}$. The second example is the union of $\mathcal{S}_b$ and $\mathcal{S}_t$ that we denote by $\mathcal{S}_{bt}$. This category is also clearly geometric over $\mathcal{S}$. The third example is $\mathcal{S}_a$, the subcategory of strong spaces $(X, \to)$, where X is in $\mathcal{S}$ and $\to$ is a weakly boolean implication, together with the strong space morphisms that U maps into $\mathcal{S}$. It is not trivial but one can show that $\mathcal{S}_a$ is also geometric over $\mathcal{S}$.

Definition 5. A subcategory $\mathcal{S}$ of **Top** is called *local* if it has at least one non-empty object and it is closed under all embeddings, i.e., for any space X in $\mathcal{S}$ and any embedding $f : Y \to X$, both Y and f belongs to $\mathcal{S}$. A space X is called *full* in $\mathcal{S}$ if it has X as an object and all maps into X as its maps.

The following theorem provides a characterization for all geometric categories over local subcategories of **Top**:

Theorem 6. *Let $\mathcal{S}$ be a local subcategory of* **Top** *with a terminal object:*

(i) *If $\mathcal{S}$ has at least one non-locally-indiscrete space, then the only geometric category over $\mathcal{S}$ is $\mathcal{S}_t$.*

(ii) *If $\mathcal{S}$ only consists of locally-indiscrete spaces, includes a non-indiscrete space and a full discrete space with two points, then the only geometric categories over $\mathcal{S}$ are the four distinct categories $\mathcal{S}_t$, $\mathcal{S}_b$, $\mathcal{S}_{bt}$ and $\mathcal{S}_a$.*

(iii) *If $\mathcal{S}$ only consists of indiscrete spaces, then the only geometric categories over $\mathcal{S}$ are the three distinct categories $\mathcal{S}_t$, $\mathcal{S}_b$, and $\mathcal{S}_{bt}$.*

As a special case, we can see that there is only one geometric category over the whole category **Top**, namely the one with the trivial implications.

Corollary 7. $\mathbf{Top}_t$ *is the only geometric category over* **Top**.

Therefore, one can conclude that there is no non-trivial and fully-geometric notion of implication.

Substructural Logics weaker than Commutative Lambek Calculus*

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Lambek Calculus [1] is a substructural logic that omits all the typical structural rules. The system is derived from its origins and can be understood as an algebraic structure, typically known as residuated monoid, and occaionally referred to as a semigroup [2]. The standard Lambek calculus is a system where only the associative law holds, as shown in Definition 1.

From the perspective of substructural logics, research has been conducted on the proof-theoretic and algebraic properties of this system by adding/ removing structural rules such as weakening, contraction, and exchange. Building on the standard Lambek calculus, researchers have identified a weaker system that incorporates only one structure rule. This paper focuses on a system that employs solely the exchange rule $\Gamma, \alpha, \beta, \Sigma \vdash \gamma \Longrightarrow \Gamma, \beta, \alpha, \Sigma$ and demonstrates the existence of a countably infinite series of logics $L_e^1, L_e^2, L_e^3, \ldots, L_e^n, \ldots$, shown in Definition 2, and their closure L_e^*, shown in Definition 3, between the standard system and the system with only exchange.

We discuss the logical systems L_e^n and L_e^*, and their relationship with L and L_e, focusing on modifications to the introduction rules for / and \ without explicitly adding the exchange rule. The analysis argues that L_e^n adn L_e^* aer fundamentally different logical system, both from each other and from L and L_e (Theorem 1). It highlights that L_e^n is stronger than L but equal to or weaker than L_e, as evidenced by the number of provable sequents; namely, the rule $/\mathrm{L}^{\mathrm{k}}$ and $\backslash\ \mathrm{L}^{\mathrm{k}}$ (and consequently $/^*$ and $\backslash\ \mathrm{L}^*$) are provable in L_e but not in L. Additionally, it is natural to consider the commutative Lambek calculus as possessing the algebraic structure of a commutative residuated monoid.

The standard Lambek calculus, which is inherently non-commutative, is characterized by the algebraic structure of a residuated monoid. Furthermore, the commutative Lambek calculus can also naturally be considered to have the algebraic structure of a commutative residuated monoid. However, the above-mentioned L_e^n and L_e^* do not fit into either algebra. We are exploring the translation of these systems into the algebraic structure of operads and plan to detail these effors in future work.

Definition 1 (Lambek Calculus L)**.** *The Lambek calculus L is a system of sequent calculus defined solely by the following inference rules. In particular, / and \ correspond to implications.*

$$\frac{}{\alpha \vdash \alpha}\,\mathrm{Ax} \qquad \frac{\Gamma \vdash \alpha \qquad \Sigma, \alpha, \Delta \vdash \beta}{\Sigma, \Gamma, \Delta \vdash \beta}\,\mathrm{Cut} \qquad \frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \beta/\alpha}\,/\mathrm{R} \qquad \frac{\alpha, \Gamma \vdash \beta}{\Gamma \vdash \alpha\backslash\beta}\,\backslash\mathrm{R}$$

$$\frac{\Gamma \vdash \alpha \qquad \Sigma, \beta, \Delta \vdash \gamma}{\Sigma, \beta/\alpha, \Gamma, \Delta \vdash \gamma}\,/\mathrm{L} \qquad \frac{\Gamma \vdash \alpha \qquad \Sigma, \beta, \Delta \vdash \gamma}{\Sigma, \Gamma, \alpha\backslash\beta, \Delta \vdash \gamma}\,\backslash\mathrm{L}$$

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Definition 2 (Mildly Commutative Lambek Calculus L_e^n)**.** *Mildly Commutative Lambek Calculus L_e^n is defined by the left introduction rules:* $/\mathrm{L}^0$,$\backslash\mathrm{L}^0$,$/\mathrm{L}^1$,$\backslash\mathrm{L}^1$, $/\mathrm{L}^2$,$\backslash\mathrm{L}^2$, *…,* $/\mathrm{L}^\mathrm{n}$,$\backslash\mathrm{L}^\mathrm{n}$*, and L's standard rules:* Ax,Cut,/R,\R*. All introduction rules* $/\mathrm{L}^\mathrm{k}$ *and* $\backslash\mathrm{L}^\mathrm{k}$ *are defined as follows.*

$$\frac{\Gamma \vdash \alpha \qquad \Sigma, \beta, \delta_1, \delta_2, \ldots, \delta_k, \Delta \vdash \gamma}{\Sigma, \beta/\alpha, \underbrace{\delta_1, \delta_2, \ldots, \delta_k}_{\text{k skip}}, \Gamma, \Delta \vdash \gamma}\ /\mathrm{L}^\mathrm{k} \qquad \frac{\Gamma \vdash \alpha \qquad \Sigma, \sigma_1, \sigma_2, \ldots, \sigma_k, \beta, \Delta \vdash \gamma}{\Sigma, \Gamma, \underbrace{\sigma_1, \sigma_2, \ldots, \sigma_k}_{\text{k skip}}, \alpha\backslash\beta, \Delta \vdash \gamma}\ \backslash\mathrm{L}^\mathrm{k}$$

Definition 3 (Mildly Commutative Lambek Calculus L_e^*)**.** *Mildly Commutative Lambek Calculus L_e^* is defined by the rules:* $/\mathrm{L}^*$,$\backslash\mathrm{L}^*$*, and L's standard rules:* Ax,Cut,/R,\R.

$$\frac{\Gamma \vdash \alpha \qquad \Sigma, \beta, \Theta, \Delta \vdash \gamma}{\Sigma, \beta/\alpha, \Theta, \Gamma, \Delta \vdash \gamma}\ /\mathrm{L}^* \qquad \frac{\Gamma \vdash \alpha \qquad \Sigma, \Theta, \beta, \Delta \vdash \gamma}{\Sigma, \Gamma, \Theta, \alpha\backslash\beta, \Delta \vdash \gamma}\ \backslash\mathrm{L}^*$$

Theorem 1. *L_e^{n-1} is weaker than L_e^n and L_e^* because the following sequent is not provable in L_e^{n-1}, but is provable in L_e^n and L_e^*. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}, \beta$ be atomic formulas. Then,*

$$\alpha_1, \underbrace{\alpha_2, \ldots, \alpha_{n+1}}_{n\ \ skip}, \alpha_1\backslash(\alpha_2\backslash(\ldots\backslash(\alpha_n\backslash(\alpha_{n+1}\backslash\beta))\ldots)) \vdash \beta.$$

Proof. First, we prove the case when $n = 1$; *i.e.*, $\alpha_1, \alpha_2, \alpha_1\backslash(\alpha_2\backslash\beta) \vdash \beta$ is provable in L_e^1 and L_e^* but not in L. The sequent is provable in L_e^1 and L_e^* by the rule $\backslash\mathrm{L}^1$. Furthermore, the sequent is not provable in L because the exhaustive proof search is halted by the cut elimination theorem. Next, we prove the case when $n = 2$; *i.e.*, $\alpha_1, \alpha_2, \alpha_3, \alpha_1\backslash(\alpha_2\backslash(\alpha_3\backslash\beta)) \vdash \beta$ is provable in L_e^2 and L_e^* but not in L_e^1. Similarly, we can prove that the sequent is provable in L_e^2 and L_e^* by the rules $\backslash\mathrm{L}^2$ and $\backslash\mathrm{L}^1$. Unlike the standard Lambek calculus, there is no cut elimination theorem in the systems L_e^n. However, we can prove that the sequent is not provable in L_e^1 by analyzing the proof search. Accordingly, we can prove the remaining cases in the same manner. Thus, we conclude that L_e^{n-1} is weaker than L_e^n and L_e^*. ☐

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Modal completeness for general scattered spaces

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Modal logic admits two, by now classical, topological semantics. One is given by interpreting the modal diamond $\Diamond$ as the closure, and the other by interpreting $\Diamond$ as the derived set[1]. We refer to [2] for a thorough overview of these semantics. A celebrated result for the closure semantics is the McKinsey and Tarski theorem stating that the modal logic $\mathsf{S4}$ is sound and complete with respect to any dense-in-itself metrizable space, in particular, any Euclidean space [9]. A landmark result for the derivative semantics is the Abashidze-Blass theorem stating that the modal logic $\mathsf{GL} = \Box(\Box p \to p) \to \Box p$ is sound and complete with respect to any ordinal $\alpha \geq \omega^\omega$ with the standard interval topology [1, 5] (See also: [4]). Earlier Esakia [7] showed that GL is sound and complete with respect to the class of scattered spaces. Recall that a topological space is *scattered* if its every non-empty subset contains a point isolated in that subset. It is easy to verify that each ordinal is a scattered space with respect to the order topology.

In modal logic, general Kripke frames constitute an important generalization of Kripke semantics. A *general Kripke frame* is a triple (X, R, A), where $A \subseteq \mathcal{P}(X)$ is a modal subalgebra of the powerset algebra. In a general frame formulas are evaluated in the algebra A. A Kripke frame can be seen as a general frame, where $A = \mathcal{P}(X)$. It is well known that, unlike Kripke semantics, every modal logic is sound and complete with respect to its general Kripke frames [6].

Similarly to Kripke frames one can consider general topological spaces for both the closure and derived set semantics. A general (topological) c-space is a pair (X, A), where X is a topological space and A is a modal subalgebra of $(\mathcal{P}(X), c)$, where $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is the topological closure. Like in general Kripke frames, in general c-spaces, formulas are evaluated in the algebra A. Bezhanishvili et al. [3] show that the McKinsey and Tarski theorem can be extended to all connected extensions of $\mathsf{S4}$ by considering general c-spaces. In particular, they showed that for every extension $L \supseteq \mathsf{S4}$, such that L is the logic of a connected $\mathsf{S4}$-algebra, there is a general c-space $(\mathbb{R}, A)$ over the real line $\mathbb{R}$ such that L is sound and complete for $(\mathbb{R}, A)$.

General topological spaces for the derived set semantics have been considered in [8] for provability logics with countably many modal operators and more recently in [10] where it was shown that the bimodal provability logic GLB is sound and complete with respect to general bi-topological spaces.

In this abstract we combine these two approaches. We will consider general topological spaces for the derived set semantics over ordinal spaces and we will prove a generalization of the Abashidze-Blass theorem for these spaces, in the same way [3] proved a generalized version of the McKinsey and Tarski theorem for general spaces over the real line.

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*speaker

[1] Recall that the derived set $d(U)$ of U consists of all those points x such that every open neighbourhood of x intersects $U \setminus \{x\}$.

Definition 1. *A general d-space is a pair (X, A) with X a topological space and $A \subseteq \mathcal{P}(X)$ a modal subalgebra of $(\mathcal{P}(X), d)$, where $d : \mathcal{P}(X) \to \mathcal{P}(X)$ is the derived set operator.*

A valuation in a general d-space is a map from propositional variables to A, which is extended to all formulas in a standard way, mapping $\Diamond\varphi$ to $d[\![\varphi]\!]$. Note that every general d-space has a least subalgebra, the d-algebra generated by $\varnothing$. We call a general d-space (X, A) where X is a scattered space and A is the least d-subalgebra of $(\mathcal{P}(X), d)$, a *least scattered d-space.* Recall that $\mathsf{GL.3} = \mathsf{GL} + (\Diamond p \wedge \Diamond q \to \Diamond(p \wedge q) \vee \Diamond(p \wedge \Diamond q) \vee \Diamond(q \wedge \Diamond p))$.

Theorem 1. *Let (X, A) be a least scattered d-space. Then (X, A) validates* $\mathsf{GL.3}$.

Recall that Kripke frames of $\mathsf{GL.3}$ are linear dually well-founded frames (i.e., linear GL-frames)[6]. The above result can be extended to a completeness of all extensions of $\mathsf{GL.3}$.

Theorem 2. *For every extension $L \supseteq \mathsf{GL.3}$ there exists an ordinal $\alpha \leq \omega^\omega$ and a least scattered d-space (α, A) over α such that L is the logic of (α, A).*

The above theorem can in fact be generalized to a much larger class.

Theorem 3. *Let $L \supseteq \mathsf{GL}$ be a Kripke complete extension of GL. Then there exists a countable ordinal α and a general scattered d-space (α, A) over α, such that L is the logic of (α, A). Furthermore, if L enjoys the finite model property, then $\alpha \leq \omega^\omega$.*

We leave it as an open problem whether any extension of GL (i.e., not Kripke complete ones) is complete with respect to a class of general scattered d-spaces. Another interesting direction for future research is to study least general d-spaces beyond scattered spaces and to investigate completeness of modal logics, not necessarily of extensions of GL, with respect to general topological d-spaces.

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Domains arising in operator algebras

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Abstract

Continuous dcpos naturally arise in the study of operator algebras as families of certain modules, equipped with the partial order given by inclusion. These domains play a crucial role in noncommutative topology.

I will briefly explain the construction of these domains, give some important examples, and then present a recent structure result showing that in many cases the domains are semilattices.

Background

In 2008, a surprising connection between domain theory and operator algebras was discovered [CEI08], see also [Kei17]: The isomorphism classes of countably generated Hilbert modules over a C^*-algebra A, equipped with the partial order induced by the inclusion of closed submodules, is a domain (a continuous dcpo). Moreover, direct sum of modules defines an abelian addition on this domain, turning it into a domain semigroup. The construction goes back to Cuntz [Cun78], and the said domain semigroup is therefore called the *Cuntz semigroup* $\mathrm{Cu}(A)$.

A C^**-algebra* is a norm-closed $*$-algebra of operators on a Hilbert space. These are often thought of as noncommutative topological spaces, since a C^*-algebra is commutative if and only if it is isomorphic to $C(X) = \{f : X \to \mathbb{C} | f \text{ continuous}\}$ for some compact, Hausdorff space X.

For many C^*-algebras, the Cuntz semigroup can be computed explicitly. For example, if $A = \mathbb{C}$, then countably generated Hilbert modules are nothing but separable Hilbert spaces, and these are characterized by their dimension, which gives

$$\mathrm{Cu}(\mathbb{C}) \cong \overline{\mathbb{N}} := \{0, 1, 2, 3, \ldots, \infty\}.$$

For the C^*-algebra $A = C([0, 1])$, every countably generated Hilbert modules is a bundle of separable Hilbert spaces over the base space $[0, 1]$, with the dimension of the fibers varying lower-semicontinuously, which gives

$$\mathrm{Cu}(C([0, 1])) \cong \mathrm{Lsc}([0, 1], \overline{\mathbb{N}}).$$

Other examples of domain semigroups arising as the Cuntz semigroup of a C^*-algebra are $[0, \infty]$ (with the usual addition and order), and the semigroup $\mathrm{LAff}(K)_{++} \cup \{0\}$ of lower-semicontinuous, affine functions $K \to (0, \infty]$ for a Choquet simplex K.

Since 2008, domain semigroups have been studied extensively in the context of C^*-algebras. In particular, the author and collaborators have shown that domain semigroups form a closed, symmetric monoidal category [APT18, APT20].

A C^*-algebra is said to have *stable rank one* if its invertible operators are norm-dense, a property that is known to be equivalent to the ring-theoretic notion of Bass stable range one.

Stable rank one is a finiteness assumption that is automatically satisfied in many situations of interest. Recently [APRT22], a deep structure result was shown for the domain semigroups arising from such C^*-algebras:

Theorem 1. *Given a C^*-algebra A with stable rank one, the Cuntz semigroup* $\mathrm{Cu}(A)$ *has the Riesz interpolation property, that is, whenever countably generated Hilbert modules E_1, E_2 and F_1, F_2 satisfy*

$$E_j \hookrightarrow F_k$$

for $j = 1, 2$ and $k = 1, 2$, then there exists a countably generated Hilbert module G such that

$$E_j \hookrightarrow G \hookrightarrow F_k$$

for $j = 1, 2$ and $k = 1, 2$.

This shows that for two elements x and y in $\mathrm{Cu}(A)$, the set of lower bounds for $\{x, y\}$ is upward directed and therefore has a supremum, which means that the infimum $x \wedge y$ exists. Thus, the Cuntz semigroup not only has the structure of a domain semigroup, but it is also a continuous inf-semilattice.

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Canonical Approximations of Modal Logics

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Recall that a modal logic Λ is called *canonical* when its variety of algebras is closed under taking canonical extensions [2, Definition 5.44]. It is well known that this is equivalent to being $\mathcal{D}$-persistent, i.e. having the property that for every descriptive frame of Λ, the underlying Kripke frame is also a Λ-frame [2, Proposition 5.85].

The most important property of canonical logics is that they are *strongly Kripke complete*. As such, canonicity is a major tool for establishing Kripke completeness for modal logics. In addition, many logics of interest are canonical. Sahlqvist's completeness theorem states that every logic axiomatised by Sahlqvist formulas is canonical [2, Theorem 4.42], thus establishing a convenient syntactic description for a large subclass of canonical logics. Moreover, the Fine-van Benthem theorem states that every logic characterised by an elementary class of frames is canonical [3, Theorem 10.19].

In spite of these results, several well-known modal logics are not canonical, most notably the McKinsey logic **K.1** (or **KM**), the Gödel-Löb logic **GL** and Grzegorczyk's logic **Grz** [3, Section 6.2] [4].[1] In addition, several common extensions, such as **Grz.2** and **Grz.3**, are not canonical.

We are interested in finding closest canonical "approximations" for (non-canonical) normal modal logics.

Approximations. Let $\mathsf{NExt}(\mathbf{K})$ denote the set of all normal modal logics, and let $\mathcal{X} \subseteq \mathsf{NExt}(\mathbf{K})$ be a set of normal modal logics such that $(\mathcal{X}, \subseteq)$ forms a complete lattice. For a logic Λ not necessarily in $\mathcal{X}$, define the $\mathcal{X}$*-approximation of* Λ *from below* resp. *from above* to be

$$\mathcal{X}_{\uparrow}(\Lambda) := \bigvee \{\Lambda' \in \mathcal{X} \mid \Lambda' \subseteq \Lambda\} \qquad \text{and} \qquad \mathcal{X}_{\downarrow}(\Lambda) := \bigwedge \{\Lambda' \in \mathcal{X} \mid \Lambda \subseteq \Lambda'\}$$

respectively. Clearly, a completely analogous definition can be used in the intuitionistic setting.

When $(\mathcal{X}, \subseteq)$ is a complete *sub*lattice of $(\mathsf{NExt}(\mathbf{K}), \subseteq)$ the meet is the intersection of logics and the join is the sum, i.e. the least normal modal logic containing the union of the logics, and we obtain

$$\mathcal{X}_{\uparrow}(\Lambda) \subseteq \Lambda \subseteq \mathcal{X}_{\downarrow}(\Lambda).$$

In this case the approximation from above is the least logic in $\mathcal{X}$ extending Λ and the approximation from below is the greatest sublogic of Λ contained in $\mathcal{X}$.

Taking for $\mathcal{X}$ the set of weakly Kripke complete normal modal logics, the approximation from above is just the logic of the frame class, i.e. $\mathsf{Log}(\mathsf{Fr}(\Lambda))$. In the intuitionistic setting, [1, 5] studied approximations where the set of super-intuitionistic subframe logics and the set of super-intuitionistic stable logics are taken for $\mathcal{X}$. Canonical approximations however, have not been studied before.

[1] Recall that **K.1** is the normal modal logic axiomatised by the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$, **GL** is the logic of irreflexive conversely wellfounded frames and **Grz** the logic of reflexive conversely wellfounded frames [3, Section 3.5 and Table 4.2].

Canonical approximations. Let us write Can for the set of canonical normal modal logics. We note the following.

Theorem 1. Can *is closed under arbitrary sums and finite intersections, but not under infinite intersections. Hence it forms a complete lattice, and a sublattice of* $(\mathsf{NExt}(\mathbf{K}), \subseteq)$*, but not a complete sublattice.*

Even though the closure under intersections is stated as Problem 10.2 in [3], the proof turns out to be an easy exercise.[2]

Since Can is closed under arbitrary sums, $\mathsf{Can}_{\uparrow}(\Lambda) \subseteq \Lambda$ for every logic Λ. Interestingly, however, the dual inequality need not hold: the canonical approximation from above of a logic need not extend the logic. This is exemplified by the following theorem, which follows from the Fine-van Benthem theorem.

Theorem 2. *Let Λ be a logic that has the finite model property. Then* $\mathsf{Can}_{\downarrow}(\Lambda) = \mathsf{Can}_{\uparrow}(\Lambda)$.

Clearly this means that for non-canonical logic Λ which has the finite model property, e.g. **GL** or **Grz**, $\Lambda \not\subseteq \mathsf{Can}_{\downarrow}(\Lambda)$. In fact the canonical approximation from above of a logic can be expressed as a kind of special case of the one from below by the formula

$$\mathsf{Can}_{\downarrow}(\Lambda) = \mathsf{Can}_{\uparrow}\left(\bigcap\{\Lambda' \in \mathsf{Can} \mid \Lambda \subseteq \Lambda'\}\right).$$

Recall that over **S4**, the McKinsey axiom, denoted *.1*, corresponds to the class of frames in which every point sees a point that sees only itself. The *.2* axiom expresses the *confluence* or *Church-Rosser* property, and the *.3* axiom expresses linearity of frames [3, Section 3.5 and Table 4.2]. Using selection-based methods, we compute the canonical approximations of **Grz.2** and **Grz.3**.

Theorem 3.

(i) $\mathsf{Can}_{\downarrow}(\mathbf{Grz.2}) = \mathsf{Can}_{\uparrow}(\mathbf{Grz.2}) = \mathbf{S4.2.1}$,

(ii) $\mathsf{Can}_{\downarrow}(\mathbf{Grz.3}) = \mathsf{Can}_{\uparrow}(\mathbf{Grz.3}) = \mathbf{S4.3.1}$.

In a sense, in these two cases the canonical approximation is obtained by "just" dropping the converse wellfoundedness from the frame conditions. This raises the question whether something similar happens for other non-canonical logics, in particular **Grz** itself and the analogous extensions of **GL**.

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[2] The fact that canonicity is not preserved under infinite intersections can be seen for example by considering the logic **GL**, known to be non-canonical, which can be shown to equal the intersection of the logics $\mathbf{K4} \oplus \Box^n \bot$.

SAT-universal CNF for Łukasiewicz logic

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The proof theory of multiple-valued logics, as well as its complexity, have been deeply studied, particularly for the class of the so-called fuzzy logics. However, the study of a systematic presentation of these logics with a view to the design of efficient satisfiability solvers has received less attention. Since satisfiability is usually the main logical question addressed in instances of real-world related problems, this study is motivated both from a purely mathematical and also a more applied perspective. Finding a clausal-form like definition that would help the automatic management of the SAT question is a rather open question, which we will address in this work. We will focus here in SAT as the problem of determining, for a given formula, whether there is an assignment making that formula true (sometimes called strong SAT), as opposed to other definitions related to assigning a particular value to the formula.

It is immediate that SAT for Gödel and Product logics is equal to that of classical logic (see eg. [3]), but the Łukasiewicz logic case offers deeper challenges. In the literature, we find studies on purely syntactical clausal forms for Łukasiewicz logics for instance in [4] and [2]. While the first one addresses only a subclass of Łukasiewicz formulas, the second offers a definition of a clausal form which is universal for SAT, but seems of limited use when attempting to design a resolution-like algorithm.

We propose a definition of clausal form for Łukasiewicz logic that is universal for SAT and whose structure offers a high potential, since the many-valued operators (namely, the non lattice ones) are applied to single literals.

Definition 1.1. We let monadic Łukasiewicz formulas be the formulas build with the language $\oplus, \odot$ and a single literal.

For instance, $((\neg x)^3 \oplus (\neg x)) \odot (3\neg x)$ is a monadic Łukasiewicz term, while $x \oplus y$ or $x \odot \neg x$ are not.[1]

Definition 1.2. A formula φ is in Ł-SAT conjunctive normal form if it has the structure

$$\bigwedge_{i\in I} \bigvee_{j\in J} t_{i,j}(x_{i,j})$$

for t monadic Łukasiewicz formulas.
We denote by Ł-SAT$_{CNF}$ to the set of formulas in Ł-SAT conjunctive normal form.

We can define a mapping $\sigma\colon Fm \to$Ł-SAT$_{CNF}$ in such a way that the following result holds:

Theorem 1.3. *Let φ be a Łukasiewicz formula. Then φ is SAT if and only if $\sigma(\varphi)$ is SAT.*

The proof and construction rely in several known results about Łukasiewicz logic, namely:

Lemma 1.4 (from [1])**.** *φ is SAT in Ł if and only if it is SAT in MV_n for some $n \leqslant (\frac{\sharp\varphi}{n})^n$, for $\sharp\varphi$ the number of apparitions of variables in φ and n the number of different variables in φ.*

[1] By l^n or nl we mean the usual application of the Łukasiewicz product or sum n times.

This implies that φ is SAT in Ł if and only if it is SAT in MV_{k_φ}, for $k_\varphi = mcm(\{p : p \leqslant \frac{\sharp\varphi}{n})^n, p \text{ prime}\})$.

Lemma 1.5 (Existence of so-called Ostermann terms, from [5])**.** *Let $a \in [0,1]$ be a finite sum of inverses of powers of 2. Then there is a formula from $[0,1]$ in $[0,1]$ in one free variable $\tau_a(x)$, such that*

1. *$\tau_a(x) = 1$ if and only if $x \geqslant a$,*
2. *$\tau_a(x)$ is a composition of $y \odot y$ and $y \oplus y$.*

We do not detail the construction of σ here for lack of space, but the sketch of the definition and proof of universality is as follows.

Let us denote by $\mathbb{D}$ the finite sums of inverse powers of 2 belonging to $[0,1]$, as in Lemma 1.5. It is easy to check that, given any n, we can chose some finite $\mathbb{D}_n \subset \mathbb{D}$ such that $0, 1 \in \mathbb{D}_n$ and for every $i/n, (i+1)/n \in MV_n$ (for $i < n$) there is a single $d_i \in \mathbb{D}_n$ for which $i/n < d_i < i+1/n$, and such that no other element belongs to $\mathbb{D}_n$. Furthermore, an involutive negation can be defined over them in the obvious way (namely, $\sim d_i = d_{n-i-1}$), as well as two suitable notions of (closed) product between them (roughly speaking, the top one, and the bottom one). Using these ideas, in combination with the above completeness for SAT with respect to a single finite algebra, we can define constructively the translation σ relying in the possibility to split each implication ($a \to b = 1$ if and only if, for any element x in $\mathbb{D}_{k_\varphi}$, either $a \leqslant x$ or $x \leqslant b$). The involutive negation (both over the elements of the algebra and over $\mathbb{D}_{k_\varphi}$), when used carefully, allows us to address both inequalities as the previous ones, leading to a total splitting of the formulas in Ostermann terms over the elements in $\mathbb{D}_{k_\varphi}$ applied to the literals arising from the variables in φ. The distributivity of MV algebras allows to conclude the final form as CNF.

We will also present a resolution method complete with respect to the presented forms, which needs of a finite number of rules to produce an assignment satisfying the formula. The fact that the outermost level is that of classical CNF, and that the multi-valuedness is limited to single variables makes this forms amenable to be solved either in the previous way or modeled with tools like MIP or relying, for the outermost level of the solving algorithm, in efficient classical SAT solvers. Furthermore, while the bound for finite satisfiability under Lemma 1.4 for the translated formula would be very high, a refinement of our Theorem, following from the proof itself, is that φ is SAT if and only if $\tau(\varphi)$ is SAT in MV_{k_φ}.

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A bi-equivalence between topoi with enough points and a localisation of topological groupoids

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Our contribution. In this presentation, we will demonstrate a bi-equivalence

$$\mathbf{Topos}^{\text{iso}}_{w.e.p.} \simeq [\mathfrak{W}^{-1}]\mathbf{LogGrpd},$$

where:

- $\mathbf{Topos}^{\text{iso}}_{w.e.p.}$ is the bi-category of topoi with enough points, geometric morphisms, and natural isomorphisms,
- $\mathbf{LogGrpd} \subseteq \mathbf{TopGrpd}$ is a bi-subcategory of the bi-category of topological groupoids – the bi-category of *logical groupoids*,
- and $\mathfrak{W}$ is a *left* bi-calculus of fractions on $\mathbf{LogGrpd}$.

Background on localic representations of topoi. It is often remarked that Grothendieck topoi are a generalisation of topological spaces, in their point-free incarnation, where 'points can have non-trivial isomorphisms'. As proved by Joyal and Tierney [4], every topos $\mathcal{E}$ is *represented* by some *localic* groupoid $\mathbb{X}$, in the sense that $\mathcal{E}$ is equivalent to the *topos of sheaves* $\mathbf{Sh}(\mathbb{X})$.

In [5], Moerdijk demonstrates that a geometric morphism $\mathbf{Sh}(\mathbb{X}) \xrightarrow{f} \mathbf{Sh}(\mathbb{Y})$ is induced by a cospan

$$\begin{tikzcd} \mathbb{W} \arrow[r] \arrow[d] & \mathbb{Y} \\ \mathbb{X} & \end{tikzcd}$$

of homomorphism of localic groupoids, and moreover a bi-equivalence

$$\mathbf{Topos}^{\text{iso}} \simeq \mathbf{ECG}[\Sigma^{-1}] \tag{1}$$

between the bi-category of topoi (with only invertible 2-cells) and a localisation on the *right* of a bi-subcategory $\mathbf{ECG} \subseteq \mathbf{LocGrpd}$ of localic groupoids (where the details of the bi-category fractions are handled in a paper by Pronk [6]).

Topological representation of topoi. Since any topos with enough points can be represented by a *topological* groupoid (see [3]), it is natural to wonder whether a version of the bi-equivalence (1) exists where localic groupoids are replaced by topological groupoids. However, we can demonstrate that:

Proposition 1. *For any bi-subcategory $\mathcal{C} \subseteq$ **TopGrpd**, and any* right *bi-calculus of fractions Σ on $\mathcal{C}$,*

$$\mathbf{Topos}^{\text{iso}}_{w.e.p.} \not\simeq \mathcal{C}[\Sigma^{-1}].$$

This motivates our adoption of a *left* bi-calculus of fractions in the result:

Theorem 2. *There is a bi-equivalence* $\mathbf{Topos}^{\text{iso}}_{w.e.p.} \simeq [\mathfrak{W}^{-1}]\mathbf{LogGrpd}$.

An application to model theory. A classical result of model theory asserts that an *atomic/ω-categorical* theory $\mathbb{T}$ is characterised, up to bi-interpretability, by the topological automorphism group $\mathrm{Aut}(M)$ of its unique countable model (see [1]), i.e. given atomic theories $\mathbb{T}_1$ and $\mathbb{T}_2$ with countable models M and N,

$$\mathbb{T}_1, \mathbb{T}_2 \text{ are bi-interpretable} \iff \mathrm{Aut}(M) \cong \mathrm{Aut}(N).$$

Recently, Ben Yaacov has shown that *any* theory is characterised up to bi-interpretability by a topological groupoid [2]; however, his groupoid is *not* a groupoid of models for the theory.

From our bi-equivalence, we will deduce a groupoidal extension of the classical Ahlbrandt-Ziegler result: given theories $\mathbb{T}_1, \mathbb{T}_2$,

$$\mathbb{T}_1, \mathbb{T}_2 \text{ are Morita equivalent} \iff \mathbb{X}, \mathbb{Y} \text{ are weakly equivalent},$$

where $\mathbb{X}$ and $\mathbb{Y}$ are representing topological groupoids of models for the classifying topoi of $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively.

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Effective Descent Morphisms in the Dual Categories of ((Compact) Hausdorff) Topological Spaces, Banach spaces, and Some Other Concrete Categories

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Descent theory begins with Grothendieck's remarkable work [4]. The context it uses consists of a fibration of categories over a category $\mathcal{C}$, and a morphism $p : E \to B$ in $\mathcal{C}$; the morphism p is called an effective descent morphism (resp. descent morphism) if the canonical morphism from the fibre over B to a certain category of descent data that is described in terms of the fibre over E is an equivalence of categories (resp. is full and faithful). Usually one restricts, however, to the case of basic fibration, where the fibres over B and E are comma categories $\mathcal{C}/B$ and $\mathcal{C}/E$, respectively. Later, independently J. Beck (unpublished), and J. Bénabou and J. Roubaud [1] worked out the monadic approach to descent theory. In particular, it was proved that in the case of basic fibration, a morphism p is an effective descent morphism (resp. descent morphism) if and only if the pullback functor $p^* : \mathcal{C}/B \to \mathcal{C}/E$ is monadic (resp. premonadic). In many concrete situations, describing such morphisms is a highly non-trivial problem, with many publications of various authors devoted to it. For instance, effective descent morphisms in the category of topological spaces were characterized by Reiterman and Tholen [6], Clementino and Hofmann [2], Clementino and Janelidze [3]. Effective descent morphisms in the category of Hausdorff spaces were described by Clementino and Janelidze [3].

In [10], we reduced the problem whether all descent morphisms are effective in a category with a factorization system $(\mathbb{E}, \mathbb{M})$ (with $\mathbb{M} \subseteq M$) to the simpler one. Namely, we have shown that this problem is equivalent to the one obtained from it by replacing "all descent morphisms" by "all descent morphisms from $\mathbb{E}$", and by replacing arbitrary descent data in the definition of an effective descent morphism by descent data of a certain kind. The goal of this talk is to present some applications of this simplification. Below we use codescent for descent in dual categories.

A new proof of the following fact is found: *every regular monomorphism in the category of topological spaces (i.e., an embedding) is an effective codescent morphism.* Note that, initially, this fact was proved by Mantovani in a different way (unpublished). The third proof of this fact arises from the results on effective descent morphisms in topological categories given in [10]. The similar results are obtained for the categories of uniform spaces, proximity spaces, and some other topological categories.

The problem when a functor reflects effective descent morphisms is simplified: as different from the similar results known earlier, we require a functor to preserve not all pullbacks, but pullbacks of $\mathbb{E}$-morphisms. This enables us to study effective codescent morphisms in the duals of several more categories of topological nature since, in topology, there is a number of forgetful functors which do not preserve pushouts, but preserve pushouts of some monomorphisms. In particular, the following statement is obtained: *every codescent morphism in the category of Hausdorff spaces is effective.*

Note that not every regular monomorphism in the category of Hausdorff spaces (i.e. a closed embedding) is a codescent morphism (Kelly [5]). We gave the following statement. Before we formulate it, recall that subsets U_1 and U_2 of a topological space B are called completely separable if there exists a continuous mapping from B to the closed interval $[0, 1]$ such that $f(U_1) = \{0\}$ and $f(U_2) = \{1\}$.

For a closed embedding $p : B \rightarrowtail E$ in the category of Hausdorff spaces and, for the following conditions, one has (i)⇒(ii)⇐(iii). If E is regular and B is compact, then (i) and (ii) are equivalent. If, again, E is regular and each two disjoint open subsets of B are completely separable, then all three conditions are equivalent: (i) p is an effective codescent morphism; (ii) for any completely separable open subsets U_1 and U_2 of B, there exist disjoint open subsets V_1 and V_2 of E such that $U_1 = B \cap V_1$ and $U_2 = B \cap V_2$; (iii) for any disjoint open subsets U_1 and U_2 of B, there exist disjoint open subsets V_1 and V_2 of E such that $U_1 = B \cap V_1$ and $U_2 = B \cap V_2$.

Further, the following statements are obtained: *every monomorphism in the category of compact Hausdorff spaces (i.e., an injective continuous mapping) is an effective codescent morphism. Every regular monomorphism (i.e., an isometric embedding) in the category of Banach spaces (with linear contractions) is an effective codescent morphism.*

Effective descent morphisms in topological categories are studied. The obtained results imply that *if $\mathcal{V}$ be a Mal'cev variety of universal algebras, then every regular epimorphism (i.e., a continuous open surjective homomorphism) is an effective descent morphism in the category of topological $\mathcal{V}$-algebras.*

Finally, note that, with the aid of the above-mentioned simplification of the descent problem, effective codescent morphisms are characterized in some varieties of universal algebras [9], [11], [7], [8], [12].

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An extension of Stone duality to T_0-spaces and sobrifications

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De Vries [5] developed a renowned de Vries duality between the category **KHaus** of compact Hausdorff spaces with continuous maps and the category **Dev** of de Vries algebras with de Vries morphisms. In [1], Bezhanishvili and Harding extended de Vries duality to stably compact spaces by replacing the category of de Vries algebras with regular proximity frames. They established a de Vries duality between the category **StKSp** of stably compact spaces with proper maps and the category **RPrFrm** of regular proximity frames with proximity morphisms.

In [7], Smyth generalized the compactifications of completely regular spaces to the stable compactifications of T_0-spaces. He showed that the equivalence classes of stable compactifications of a given T_0-space form a poset. The largest element, named as the *Smyth compactification* in [3], is a generalization of the Stone-Čech compactification.

It is well known that the Stone-Čech compactification yields a reflector β : **CReg** $\to$ **KHaus** between the category **CReg** of completely regular spaces with continuous maps and the category **KHaus**. That is, **KHaus** is a full reflective subcategory of **CReg**. In [4], by introducing the category **Comp** of compactifications of completely regular spaces, Bezhanishvili, Morandi and Olberding proved that the category **CReg** is equivalent to the full subcategory **SComp** of **Comp** consisting of Stone-Čech compactifications of completely regular spaces. To develop the de Vries duality for completely regular spaces, they introduced the category **DeVe** of de Vries extensions, and built the dual equivalence between the categories **Comp** and **DeVe**. Under this duality, the full subcategory **MDeVe** of **DeVe** comprising maximal de Vries extensions was placed into duality with the category **SComp** of Stone-Čech compactifications of completely regular spaces.

Bezhanishvili and Harding developed two methods to establish the duality for T_0-spaces. On the one hand, in [3], by considering the category **StComp** of stable compactifications of T_0-spaces, they proved that the full subcategory **Smyth** of **StComp** composed of smyth compactifications of T_0-spaces is equivalent to the category $\mathbf{Top_0}$ of T_0-spaces. To extended the de Vries duality of stably compact spaces to T_0-spaces, they introduced the category **RE** of Raney extensions and established a duality between the categories **StComp** and **RE**. Thus it yielded a duality between the category $\mathbf{Top_0}$ and the full subcategory **MRE** of **RE** consisting of maximal Raney extensions. On the other hand, in [2], they developed an alternate duality between the category $\mathbf{Top_0}$ and the category **RAlg** of Raney algebras.

As we all know, sober spaces are closely related to pointfree topology and logic because of the duality (*Kawahara duality*) for spatial frames (see [6]). And the category **Sob** of sober spaces is a full reflective subcategory of the category $\mathbf{Top_0}$. In this paper, instead of stably compactifications of T_0-spaces, we choose to employ sobrifications of T_0-spaces to construct a new duality for T_0-spaces. We introduce the definition of a spatial frame Raney extension as follows.

Definition 1. Let L be a spatial frame and K a Raney lattice, where Raney lattice is a completely distributive complete lattice generated by completely join-irreducible elements. A frame homomorphism $\varepsilon : L \to K$ is said to be a *spatial frame Raney extension* if it is injective and $\varepsilon(L)$ is dense in K.

Then we establish a one-to-one correspondence between spatial frame Raney extensions and sobrifications of T_0-spaces. In order to build a duality for T_0-spaces, we introduce the category of spatial frame Raney extensions and the category of sobrifications of T_0-spaces as follows.

Definition 2. The category of spatial frame Raney extensions, denoted by **SFrmRE**, is the category whose objects are spatial frame Raney extensions $\varepsilon : L \to K$ and whose morphisms are pairs (ϕ, ψ) where $\phi : L \to L'$ is a frame homomorphism, $\psi : K \to K'$ is a complete lattice homomorphism, and $\varepsilon' \circ \phi = \psi \circ \varepsilon$.

Definition 3. The category of sobrifications, denoted $\mathbf{Sob_f}$, is the category whose objects are sobrifications $s : X \to Y$ and whose morphisms are pairs (f, g) of continuous maps, and the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "s'"'] & Y' \end{tikzcd}$$

We obtain one of the main theorem of this paper.

Theorem 4. The categories **SFrmRE** and $\mathbf{Sob_f}$ are dually equivalent; and the category $\mathbf{Sob_f}$ is equivalent to the category $\mathbf{Top_0}$.

Therefore, by Theorem 4, we obtain the duality for T_0-spaces.

Theorem 5. The category $\mathbf{Top_0}$ is dually equivalent to the category **SFrmRE**.

Especially, we apply the duality for T_0-spaces to its full subcategory $\mathbf{CKTop_0}$ consisting of core-compact T_0-spaces. And we denote the category **CFrmRE** be the full subcategory of **SFrmRE** consisting of continuous frame Raney extensions, where continuous frame Raney extension is a special spatial frame Raney extension $\varepsilon : L \to K$ with L as a continuous frame. Then we obtain the following result.

Theorem 6. There is a dual equivalence between the categories $\mathbf{CKTop_0}$ and **CFrmRE**.

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Sheaf Semantics for Inquisitive Logic

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1 Introduction

Inquisitive logic [8, 3, 1] is a logic of so-called inquisitive propositions, intended to model questions in much the same way that the propositions of non-inquisitive logic model declarations. This logic has many interesting linguistic applications [2]. First-order inquisitive logic was studied in, *e.g.*, [7], and intuitionistic inquisitive logic was introduced in [10, 11].

In our talk, we provide a categorical analysis of the main mathematical features of inquisitive logic. In particular, we give a sheaf-theoretic semantics for (higher-order, intuitionistic) inquisitive logic. This subsumes as special cases the classical possible-worlds model of inquisitive logic [12], a refinement of this based on a topological space of worlds, as well as other models with a topological flavor.

It was observed in the propositional case by [9] that the language of (intuitionistic) inquisitive logic can be identified with (intuitionistic) logic, together with a geometric modality ∇ in the sense of [6], also known as a Lawvere-Tierney modality or lax modality. Inquisitive logic is then characterized by the addition of the so-called 'split' axiom.

$$\frac{\nabla\alpha \to \phi \vee \psi}{(\nabla\alpha \to \phi) \vee (\nabla\alpha \to \psi)}\ \text{Split}$$

From the inquisitive perspective, ∇ is understood as the presupposition modality, with $\nabla\alpha$ representing the declarative proposition presupposed by the inquisitive proposition α.

2 Higher-Order Semantics

To extend Holliday's insight from the propositional setting to higher-order, we must pass from Heyting algebras and nuclei to toposes and Cartesian reflectors.

Essentially since Lawvere and Tierney, it has been known that a topos $\mathcal{E}$ equipped with with a Cartesian reflector $J : \mathcal{E} \to \mathcal{E}$ interprets intuitionistic higher-order logic with a geometric modality. The Lawvere-Tierney operator $j : \Omega \to \Omega$ in $\mathcal{E}$ induced by J interprets the geometric modality ∇. The rest of the logic is interpreted standardly in $\mathcal{E}$. Our move will be to narrow down this abstract semantics in order to validate the additional axioms of inquisitive logic.

Theorem 1. *Let $(\mathbf{C}, J)$ be a site where $\mathbf{C}$ is small and cocomplete and J is canonical. Then, $\mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$, together with the sheafification $a : \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ induced by J is a model of of intuitionistic higher-order inquisitive logic.*

3 Examples

Example 2. *Let W be a set (of possible worlds). Then, the singleton injection*

$$\{\cdot\} : W \rightarrowtail \mathbf{2}^W$$

induces the adjunction

$$\{\cdot\}^* \dashv \{\cdot\}_* : \mathbf{Set}^W = \mathbf{Set}^{W^{\mathrm{op}}} \rightarrowtail \mathbf{Set}^{(\mathbf{2}^W)^{\mathrm{op}}} \quad .$$

*The composite $\{\cdot\}_*\{\cdot\}^*$ is a Cartesian reflector, and thus induces a coverage of $\mathbf{2}^W$, which is canonical. Moreover, $\mathbf{2}^W$ is small and cocomplete (*i.e.* admits small joins).*

This recovers the classical model of predicate inquisitive logic. In particular, we have $\mathrm{Sub}_{\mathbf{Set}^{(\mathbf{2}^W)^{\mathrm{op}}}}(1) \cong \mathbf{2}^{(\mathbf{2}^W)^{\mathrm{op}}}$ and $\mathrm{Sub}_{\mathbf{Set}^W}(1) \cong \mathbf{2}^W$, i.e. *the subsingletons of $\mathbf{Set}^{(\mathbf{2}^W)^{\mathrm{op}}}$ and $\mathbf{Set}^W$ correspond respectively to downwards-closed sets of subsets of W and subsets of W, which in inquisitive logic following [12] are respectively identified with inquisitive propositions and declarative propositions.*

Example 3. *Any topological space W (of possible worlds), regarded as a site, satisfies the conditions of Theorem 1. Thus, $\mathbf{Set}^{\mathcal{O}(W)^{\mathrm{op}}}$, together with the sheafification*

$$\mathbf{Set}^{\mathcal{O}(W)^{\mathrm{op}}} \xrightarrow{a} \mathrm{Sh}(W) \hookrightarrow \mathbf{Set}^{\mathcal{O}(W)^{\mathrm{op}}}$$

is a model.

In particular, we have $\mathrm{Sub}_{\mathbf{Set}^{\mathcal{O}(W)^{\mathrm{op}}}}(1) \cong \mathbf{2}^{\mathcal{O}(W)^{\mathrm{op}}}$ and $\mathrm{Sub}_{\mathrm{Sh}(W)}(1) \cong \mathcal{O}(W)$, which we might identify with answerable inquisitive propositions and verifiable declarative propositions, respectively.

The classical model of Example 2 is recovered in the case where W is discrete and thus $\mathcal{O}(W) = \mathbf{2}^W$.

Additional examples include sheaves on a locale, and, when size issues are dealt with, sheaves on an ionad [5, 4] and sheaves on a Grothendieck topos.

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