Local inconsistency lemmas and the inconsistency by cases property

Isabel Hortelano Martín

University of Barcelona, Barcelona, Spain
ihortema7@alumnes.ub.edu

The study of deduction-detachment theorems and their algebraic counterparts is a classical part of abstract algebraic logic. It is a well-known fact that a finitary protoalgebraic logic has a deduction-detachment theorem – briefly a DDT – if and only if the semilattice of compact deductive filters of every algebra of the corresponding type is dually Brouwerian (see, e.g.,[4]). The bridge theorem has algebraic consequences, which in turn have logical applications crossing back over the bridge. For instance, any finitary protoalgebraic logic satisfying a DDT is filter-distributive, and the logical counterpart of filter-distributivity is the so-called *proof by cases property*, which has been extensively studied in [2, 3, 5].

In contrast, the theory of inconsistency lemmas, or ILs, for short, has not been systematically investigated so far, with a few exceptions (see, e.g., [1, 6, 7, 8]). Raftery proved in [8] that for a finitary protoalgebraic logic to have a (global) IL amounts to the demand that the join semilattice of compact deductive filters in each algebra of the corresponding type should be dually pseudo-complemented. Subsequently, Lávička [6] introduced and studied the local and parametrized local versions in a similar fashion to the hierarchy of DDTs.

Following the terminology introduced in [6], a logic $\vdash$ is said to have a *local inconsistency lemma*–briefly a LIL–if for every $n \in \mathbb{N}$, there exists a family Ψ_n of finite sets of formulas $I(x_1, \ldots, x_n)$ such that for every $\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \subseteq Fm$,

$$\Gamma \cup \{\varphi_1, \ldots, \varphi_n\} \text{ is inconsistent in } \vdash \iff \Gamma \vdash I(\varphi_1, \ldots, \varphi_n) \text{ for some } I \in \Psi_n.$$

The corresponding algebraic counterpart is the *maximal consistent filter extension property*, or MCFEP, for short, which a logic $\vdash$ is said to have if for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and every submatrix $\langle \boldsymbol{B}, G\rangle$ of $\langle \boldsymbol{A}, F\rangle$, for every maximal $\vdash$-filter H containing G there is a $\vdash$-filter H' containing F such that $H = H' \cap B$. This result established in [6] for protonegational logics [1] translates in the framework of finitary protoalgebraic logics as the following theorem:

Theorem 1. *Let $\vdash$ be a finitary protoalgebraic logic. The following are equivalent:*

1. *$\vdash$ has the LIL;*
2. *$\vdash$ has the MCFEP and for every algebra $\boldsymbol{A}$ the deductive filter A is finitely generated;*
3. *The MCFEP holds in the algebra of formulas and $\vdash$ posseses a finite inconsistent set of formulas.*

If the family Ψ_n witnessing the LIL consists of just one set of formulas $I(x_1, \ldots, x_n)$ for each $n \in \mathbb{N}$, then $\vdash$ is said to have an IL. As a first step to determine what is necessary for a LIL to reduce to an IL, we introduce the notion of *definable maximal consistent filters* – briefly DMCF. A logic $\vdash$ has DMCF if there is a formula $\delta(x_1, \ldots, x_n)$ in the language of the first-order

[1] The class of protonegational logics is introduced in [6] as a weakening of protoalgebraicity, restricting some of its defining conditions to maximal consistent theories. Particular examples are the negation fragments of protoalgebraic logics.

predicate logic (with equality), whose only non-logical symbols are the operation symbols of $\vdash$ and a unary predicate $F(x)$, such that for every model $\langle \boldsymbol{A}, F\rangle$ of $\vdash$ and elements $a_1, \ldots, a_n \in A$,

$$A = \mathrm{Fg}^{\boldsymbol{A}}_{\vdash}(F \cup \{a_1, \ldots, a_n\}) \iff \langle \boldsymbol{A}, F\rangle \models \delta(a_1, \ldots, a_n).$$

In this case, for a finitary protoalgebraic logic $\vdash$ with an LIL and DMCF, we prove that any family Ψ_n witnessing the LIL must include a finite subset of sets of formulas for each $n \in \mathbb{N}$ such that the resulting family also witnesses the LIL for $\vdash$. However, the question of whether Ψ_n can be taken to be a singleton for every n, and obtain a global IL, is more involved.

Before answering this question, we introduce another notion: a logic $\vdash$ has the *inconsistency by cases property* (ICP) when for every nonnegative integers n, m, there exists a parameterized set $\nabla(x_1, \ldots, x_n, y_1, \ldots, y_m, \vec{z})$ of formulas such that for any set $\Gamma \cup \{\varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m\}$ of formulas, $\vec{\varphi} \vdash \vec{\varphi}\nabla\vec{\psi}$ and $\vec{\psi} \vdash \vec{\varphi}\nabla\vec{\psi}$, and whenever $\Gamma \cup \{\vec{\varphi}\}$ and $\Gamma \cup \{\vec{\psi}\}$ are inconsistent in $\vdash$, then $\Gamma \cup \{\vec{\varphi}\nabla\vec{\psi}\}$ is inconsistent in $\vdash$, where, $\vec{\varphi}\nabla\vec{\psi}$ is defined as $\bigcup\{\nabla(\vec{\varphi}, \vec{\psi}, \vec{\gamma}) : \vec{\gamma} \in Fm\}$.

It turns out that, in parallel to the connection between the proof by cases property and filter-distributivity, the corresponding bridge theorem arises between the ICP and the notion of 1-distributivity. Recall that a lattice $\boldsymbol{A}$ with 1 is said to be *1-distributive* if whenever $a \vee b = 1$ and $a \vee c = 1$, then $a \vee (b \wedge c) = 1$ for all elements $a, b, c \in A$. We obtain the following result:

Theorem 2. *Let $\vdash$ be a finitary protoalgebraic logic. The following are equivalent:*

1. *$\vdash$ has the ICP and possesses a finite inconsistent set of formulas;*
2. *For every algebra $\boldsymbol{A}$, the lattice of $\vdash$-filters of $\boldsymbol{A}$ is 1-distributive;*
3. *The lattice of theories of $\vdash$ is 1-distributive.*

Since every dually pseudo-complemented join semilattice with 1 is 1-distributive and any algebraic lattice is isomorphic to the lattice of ideals of the join semilattice of its compact elements, crossing back over the bridge to the syntactical setting, this implies that any finitary protoalgebraic logic with an IL has the ICP. Moreover, we prove that for a finitary protoalgebraic logic having a LIL witnessed by Ψ_n, the demand for the family to be directed for each $n \in \mathbb{N}$ amounts to the 1-distributivity of the logic. Consequently, a finitary protoalgebraic logic has an IL if and only if it has the MCFEP, for every algebra $\boldsymbol{A}$ the deductive filter A is finitely generated, it has DMCF and it is filter-1-distributive.

References

[1] M. A. Campercholi and J. G. Raftery. Relative congruence formulas and decompositions in quasivarieties. *Algebra Universalis*, 78(3):407–425, 2017.

[2] P. Cintula and C. Noguera. The proof by cases property and its variants in structural consequence relations. *Studia Logica*, 101:713–747, 2013.

[3] J. Czelakowski. Filter distributive logics. *Studia Logica*, 43:353–377, 1984.

[4] J. Czelakowski. Algebraic aspects of deduction theorems. *Studia Logica*, 44:369–387, 1985.

[5] J. Czelakowski. *Protoalgebraic logics*, volume 10 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2001.

[6] T. Lávička. *An abstract study of completeness in infinitary logics*. PhD thesis, Charles University, 2018.

[7] T. Lávička, T. Moraschini and J. G. Raftery. The Algebraic Significance of Weak Excluded Middle Laws. *Mathematical Logic Quaterly*, 68(1):79–94, 2022.

[8] J. G. Raftery. Inconsistency lemmas in algebraic logic. *Mathematical Logic Quaterly*, 59(6):393–406, 2013.