Epimorphisms between finitely generated algebras

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Let K be a class of similar algebras and $\mathfrak{A}, \mathfrak{B} \in \mathsf{K}$.

Definition 1. *A homomorphism $f \colon \mathfrak{A} \to \mathfrak{B}$ is an* epimorphism *in K when for every $\mathfrak{C} \in \mathsf{K}$ and every pair of homomorphisms $g, h \colon \mathfrak{B} \to \mathfrak{C}$ it holds that:*

$$g \circ f = h \circ f \text{ implies } g = h.$$

A subalgebra $\mathfrak{A} \leq \mathfrak{B}$ is called epic *in K if the inclusion $i \colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ is an epimorphism in K.*

While every surjective homomorphism is an epimorphism, the converse is not true in general. An example of a nonsurjective epimorphism in the class of rings is the inclusion map from the integers into the rationals (see, e.g., [6]).

Definition 2. *When every epimorphism in K is surjective, we say that K has the* epimorphism surjectivity property *(ES property, for short).*

Our talk will focus on a slightly weaker demand, namely, the *weak epimorphism surjectivity property* (weak ES property, for short), which requires only epimorphisms between finitely generated algebras to be surjective [5]. From a logical standpoint, the interest of the weak ES property is motivated as follows: when a quasivariety K algebraizes a logic $\vdash$, the former has the weak ES property iff the latter has the Beth definability property [1], which intuitively states that whenever an element can be uniquely characterized, then it must be definable by a term.

Our main results facilitate the detection of failures of the weak ES property in a quasivariety K. To this end, we introduced the notion of a *full subalgebra*.

Definition 3. *A subalgebra $\mathfrak{A} \leq \mathfrak{B} \in \mathsf{K}$ is* full *when it is proper, $B = \mathrm{Sg}^{\mathfrak{B}}(A \cup \{b\})$ for some $b \in B$, and for every nonidentity K-congruence θ of $\mathfrak{B}$ there exists $a \in A$ such that $\langle a, b \rangle \in \theta$.*

Using this concept, we obtained the following characterization of the weak ES property, where K_{RFSI} stands for the class of *relatively subdirectly irreducible* (RFSI, for short) members of K.

Theorem 4. *A quasivariety K has the weak ES property iff for every finitely generated $\mathfrak{B} \in \mathsf{K}$ and $\mathfrak{A} \leq \mathfrak{B}$ that is full in K one of the following conditions holds:*

1. *There are two distinct $\theta, \phi \in \mathsf{Con}_{\mathsf{K}}(\mathfrak{B})$ such that $\theta{\restriction}_A = \phi{\restriction}_A$;*
2. *There are two distinct embeddings $g, h \colon \mathfrak{B} \to \mathfrak{C}$ with $\mathfrak{C} \in \mathsf{K}_{\text{RFSI}}$ such that $g{\restriction}_A = h{\restriction}_A$.*

*Speaker.

As a consequence, we obtain a purely algebraic proof of a classical result of Kreisel, stating that every variety of Heyting algebras or implicative semilattices has the weak ES property [7, Thm. 1]. On the other hand, Theorem 4 also paves the way for the following results.

Our first theorem simplifies the task of finding a counterexample to the weak ES property in quasivarieties with a near unanimity term. This includes, for instance, all quasivarieties with a lattice reduct.

Definition 5. *A quasivariety* K *is said to have an* n-ary near unanimity term *for* $n \geq 3$ *when there exists a term* $\varphi(x_1, \ldots, x_n)$ *such that*

$$\mathsf{K} \vDash \varphi(y, x, \ldots, x) \approx \varphi(x, y, x, \ldots, x) \approx \cdots \approx \varphi(x, \ldots, x, y) \approx x.$$

Theorem 6. *A quasivariety* K *with an* n*-ary near unanimity term has the weak ES property iff every finitely generated subdirect product* $\mathfrak{A} \leq \mathfrak{A}_1 \times \cdots \times \mathfrak{A}_{n-1}$*, where* $\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-1} \in \mathsf{K}_{\mathrm{RFSI}}$*, lacks subalgebras that are full and epic in* K*.*

The next result gives a useful characterization of the weak ES property in the context of congruence permutable varieties. Notably, these include all varieties with a group reduct.

Theorem 7. *A congruence permutable variety has the weak ES property iff its finitely generated RFSI members lack subalgebras that are full and epic in* K*.*

Similar results for the ES property have been obtained by Campercholi [2, Thms. 18 and 22]. For instance, [2, Thm. 22] states that an arithmetical variety K, whose class of RFSI members is universal has the ES property iff the RFSI members of K lack proper subalgebras that are epic in K. Our methods allow us to prove a similar result for the weak ES property (namely, Theorem 7) under the sole assumption that K is congruence permutable.

Lastly, we provide a result which demonstrates that the weak ES property has a significant impact on the structure theory of quasivarieties.

Theorem 8. *Let* K *be a relatively congruence distributive quasivariety, whose class of RFSI members is closed under nontrivial subalgebras. Then the weak ES property implies that* $\mathbb{V}(\mathsf{K})$ *is arithmetical.*

As a consequence, every filtral variety with the weak ES property is a discriminator variety (see also [3]). The results of this talk have been collected in the manuscript [4].

References

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