A Discussion on Double Boolean Algebras
Extended Abstract

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In lattice theory, a *polarity* [1] is triple $\mathbb{K} := (G, M, I)$ where G and M are sets and $I \subseteq G \times M$. For any $X \subseteq G$, X^* is the set of all $m \in M$ such that gIm for all $g \in X$. For any $Y \subseteq M$, Y^+ is the set of all $g \in G$ such that gIm for all $m \in Y$. In formal concept analysis, a polarity is called a *context*. A *concept* is a pair of sets (X, Y) such that $X^* = Y$ and $X = Y^+$. The set of all concepts is denoted as $\mathcal{B}(\mathbb{K})$ and forms a complete lattice $\underline{\mathcal{B}}(\mathbb{K})$. The notion of a concept is generalized to protoconcepts and semiconcepts [4]. A *protoconcept* is a pair of sets (X, Y) such that $X^{*+} = Y^+$. A *semiconcept* is a pair of sets (X, Y) such that $X^* = Y$ or $X = Y^+$. We denote the sets of all protoconcepts and semiconcepts by $\mathcal{P}(\mathbb{K})$ and $\mathcal{H}(\mathbb{K})$, respectively. It is a straightforward observation that $\mathcal{B}(\mathbb{K}) \subseteq \mathcal{H}(\mathbb{K}) \subseteq \mathcal{P}(\mathbb{K})$. The meet ($\sqcap$) and join ($\sqcup$) operations of the complete lattice $\underline{\mathcal{B}}(\mathbb{K})$ are extended to the set of protoconcepts. Two negation operators $\neg$ and $\lrcorner$ are defined on the set $\mathcal{P}(\mathbb{K})$. With respect to the meet, join, and two negations, the set $\mathcal{P}(\mathbb{K})$ forms an algebraic structure which is called the *algebra of protoconcept*. The set of all semiconcept $\mathcal{H}(\mathbb{K})$ forms a subalgebra of the algebra of protoconcept and the subalgebra is called the *algebra of semiconcept*.

On the abstraction of the algebra of protoconcept and algebra of semiconcept, the definition of *double Boolean algebra* and *pure double Boolean algebra* are introduced. The definition of double Boolean algebra is given below.

Definition 1. [4] An algebra $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ satisfying the following properties is called a *double Boolean algebra* (dBa). For any $x, y, z \in D$,

$(1a) (x \sqcap x) \sqcap y = x \sqcap y$	$(1b) (x \sqcup x) \sqcup y = x \sqcup y$
$(2a) x \sqcap y = y \sqcap x$	$(2b) x \sqcup y = y \sqcup x$
$(3a) \neg(x \sqcap x) = \neg x$	$(3b) \lrcorner(x \sqcup x) = \lrcorner x$
$(4a) x \sqcap (x \sqcup y) = x \sqcap x$	$(4b) x \sqcup (x \sqcap y) = x \sqcup x$
$(5a) x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$	$(5b) x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$
$(6a) x \sqcap (x \vee y) = x \sqcap x$	$(6b) x \sqcup (x \wedge y) = x \sqcup x$
$(7a) \neg\neg(x \sqcap y) = x \sqcap y$	$(7b) \lrcorner\lrcorner(x \sqcup y) = x \sqcup y$
$(8a) x \sqcap \neg x = \bot$	$(8b) x \sqcup \lrcorner x = \top$
$(9a) \neg\top = \bot$	$(9b) \lrcorner\bot = \top$
$(10a) x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$	$(10b) x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
$(11a) \neg\bot = \top \sqcap \top$	$(11b) \lrcorner\top = \bot \sqcup \bot$
$(12) (x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$	

where $x \vee y := \neg(\neg x \sqcap \neg y)$ and $x \wedge y := \lrcorner(\lrcorner x \sqcup \lrcorner y)$. A quasi-order (that is reflexive and transitive) relation $\sqsubseteq$ on D is obtained as: $x \sqsubseteq y \iff x \sqcap y = x \sqcap x$ and $x \sqcup y = y \sqcup y$, for any $x, y \in D$.

Now we consider the two sets $D_\sqcap := \{x \in D \ : \ x \sqcap x = x\}$ and $D_\sqcup := \{x \in D \ : \ x \sqcup x = x\}$. A *pure* double Boolean algebra is a dBa $\mathbf{D}$ such that for $x \in D$, either $x \in D_\sqcap$ or $x \in D_\sqcup$. A dBa $\mathbf{D}$ is called *contextual* if the quasi-order becomes partial-order. Moreover, if for each $y \in D_\sqcap$ and $x \in D_\sqcup$ with $y \sqcup y = x \sqcap x$, there is a unique $z \in D$ with $z \sqcap z = y$ and $z \sqcup z = x$, $\mathbf{D}$ is called *fully contextual*.

This new algebraic structure opens up several possible research directions. In [3], we study the topological representation theorem for fully contextual dBa and pure dBa. The definition of double Boolean algebra contains a large number of axioms. However, we show that the axioms $(10a)$, $(10b)$, $(11a)$, and $(11b)$ are derivable from the remaining ones.

Theorem 1. Let $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ be an algebraic structure satisfying $(1a) - (9a)$, $(1b) - (9b)$, and 12 of Definition 1, then for all $x, y, z \in D$ the following hold.

(a) $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$ and $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$.

(b) $\neg\bot = \top \sqcap \top$ and $\lrcorner\top = \bot \sqcup \bot$.

As the name suggests, for each dBa $\mathbf{D}$, there are two underlying Boolean algebras, $\mathbf{D}_\sqcap := (D_\sqcap, \sqcap, \neg, \bot)$ and $\mathbf{D}_\sqcup := (D_\sqcup, \sqcup, \lrcorner, \top)$. Moreover, the map $r : D \to D_\sqcap,\ r(x) := x \sqcap x$ preserves $\sqcap, \neg$ and $\bot$. The map $r' : D \to D_\sqcup,\ r'(x) := x \sqcup x$ preserves $\sqcup, \lrcorner$ and $\top$. We also have two injections $e : D_\sqcap \to D, e(x) = x$ and $e' : D_\sqcup \to D, e'(x) = x$ such that $r \circ e = id_{D_\sqcap}$ and $r' \circ e' = id_{D_\sqcup}$. Therefore, for a given dBa $\mathbf{D}$, we have the following:

(a) the semigroup $(D, \sqcap, \neg, \bot)$ satisfying $(1a) - (3a), (5a) - (8a)$, $(10a)$, and 12 is a retract [2] of the Boolean algebra $\mathbf{D}_\sqcap$.

(b) the semigroup $(D, \sqcup, \lrcorner, \top)$ satisfying $(1b) - (3b), (5b) - (8b)$, $(10b)$, and 12 is a retract [2] of the Boolean algebra $\mathbf{D}_\sqcup$.

The above observation gives the following representation theorem for dBa. We will sketch its proof in the talk.

Theorem 2. Let $(B, \wedge, \neg, \bot)$ and $(B', \vee', \neg', \top')$ be two Boolean algebras. Let $r : A \rightleftharpoons B : e$ and $r' : A \rightleftharpoons B' : e'$ be two embedding-retraction pair. $\mathbf{A} := (A, \sqcap, \sqcup, \neg, \lrcorner, e'(\top'), e(\bot))$ is a universal algebra where, $x \sqcap y := e(r(x) \wedge r(y)),\ x \sqcup y := e'(r'(x) \vee r'(y)),\ \neg x := e(\neg r(x))$, and $\lrcorner x := e'(\neg' r'(x))$. Then $\mathbf{A}$ is a dBa if and only if following holds.

(a) $e \circ r \circ e' \circ r' = e' \circ r' \circ e \circ r$.

(b) $e(r(x) \wedge r(e'(r'(x) \vee r'(y)))) = e(r(x))$ and $e'(r'(x) \vee r'(e(r(x) \wedge r(y)))) = e'(r'(x))$ for all $x, y \in A$.

(c) $r(e'(\top')) = \top$ and $r'(e(\top)) = \top'$.

Moreover, every dBa can be obtained from such an embedding-retraction construction.

In [4], it is shown that $D_p := D_\sqcup \cup D_\sqcap$ forms the largest pure subalgebra $\mathbf{D}_p$ of a dBa $\mathbf{D}$. Moreover, the largest pure subalgebra plays an important role in characterizing two different dBa. In particular, we will discuss the following result.

Theorem 3. Let $\mathbf{D}$ and $\mathbf{M}$ be fully contextual dBas. Then $\mathbf{D}$ is isomorphic to $\mathbf{M}$ if and only if $\mathbf{D}_p$ is isomorphic to $\mathbf{M}_p$. Moreover, every dBa isomorphism from $\mathbf{D}_p$ to $\mathbf{M}_p$ can be uniquely extended to a dBa isomorphism from $\mathbf{D}$ to $\mathbf{M}$.

References

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