Completeness of the GL.3 provability logic for the intersection of normal measures

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Abstract

Provability logic GL is known to be sound and complete w.r.t. *scattered topological spaces*, namely spaces (X, τ) where every $A \subset X$ has an isolated point. The topological interpretation is the following: topological model is a pair $\langle (X, \tau), v \rangle$, where (X, τ) is a topological space and $v : \mathsf{Vars} \to PX$, which yields an interpretation:

- $[\![\top]\!] = X; [\![\bot]\!] = \emptyset; [\![p]\!] = v(p)$;
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$;
- $[\![\neg\varphi]\!] = X \setminus [\![\varphi]\!]$;
- $[\![\Diamond\varphi]\!] = d_\tau[\![\varphi]\!]$;

where $d_\tau A = \{x : \forall U \in \tau_i \exists y \neq x (y \in U \cap A)\}$ for each $A \subset X$, we call d_τ the derivative operator.

One can show that GL is the logic of the class of scattered spaces. Moreover, GL is known to be Kripke complete w.r.t. to finite irreflexive trees [6], hence by regarding Kripke models as topologies generated by the upsets, we are already getting topological completeness for the class of scattered spaces. Moreover, GL is topologically complete with respect to any single ordinal $\geq \omega^\omega$:

Theorem 1 (Abashidze[1], Blass[2])**.** *Consider an ordinal $\Omega \geq \omega^\omega$ with its order topology, then $Log(\Omega) = \mathsf{GL}$.*

There is a number of other peculiar yet natural examples of scattered spaces. Most of them arise in the context of set theory, so expectedly the question of completeness is independent for some of them. They are worth studying, for we can see how modal logic can reflect properties of somewhat complicated objects. Another important motivation behind this is that GLP – the polymodal generalization of GL – is Kripke incomplete and the candidates for the right topological models dwell in the realm of topologies on ordinals, hence related questions naturally emerge in the study of GLP.

A. Blass in [2] provided a characterization of topologies on Ord for which GL is sound. Although his results were formulated in terms of sequences of filters, filters and topologies can be mutually interpreted, by regarding filter on α as the set of punctured neighborhoods of α. Some natural topologies that satisfy these conditions were mentioned by A. Blass in the same

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article, namely topologies corresponding to: the end-segment filter, the club-filter, the subtle filter, the ineffable filter and the filter

$$M_\kappa = \bigcap\{U : U \text{ is a normal measure on } \kappa\}$$

For this filter we call the corresponding topology τ_U, the topology given by letting $A \subset \kappa$ be a punctured neighborhood of κ if $A \in M_\kappa$ (as well as τ'_U which is essentially the same, but relativized to pseudonormal filters). These topologies are the main subject of our study.

Golshani and Zoghifard in [3] have shown that there is a model of ZFC, where $\mathrm{Log}(\langle \mathsf{Ord}, \tau_U\rangle) = \mathsf{GL}$, provided there exists infinitely many strong cardinals. Whereas Blass has mentioned that GL can fail to be the logic of this topology, since there is a model $L[\mathcal{U}]$ – class of sets constructible from a sequence of normal ultrafilters (cf.[5], [4]). In this model a non-theorem of GL holds for τ_U, however it was open what exactly the logic of it is. Let

$$\mathsf{GL}.3 = \mathsf{GL} + \Box(\Box A \to B) \vee \Box(\Box B \wedge B \to A)$$

The main result of our investigation is:

Theorem 2. $\mathrm{Log}(\langle \mathsf{Ord}, \tau\rangle) = \mathsf{GL}.3$ *if* $V = L[\mathcal{U}]$ + *"there exists a measurable cardinal of Mitchell order* n *for each* $n < \omega$*". Where* $\tau \in \{\tau_U, \tau'_U\}$.

Moreover, we have shown

Theorem 3. $\mathrm{Log}(\langle \mathsf{Ord}, \tau'_U\rangle) = \mathsf{GL}.3$ *if* AD *holds.*

Generalization of this result is the matter of our future research.

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