Remarks on the DeMorganization of a locale

Igor Arrieta[1]

School of Computer Science, University of Birmingham, UK
i.arrietatorres@bham.ac.uk

Regular subobjects (equivalently, extremal subobjects) in the category of locales are known as sublocales, and therefore they are the point-free counterparts of classical subspaces of a space. In a topological space X, every subspace induces a sublocale of its frame of opens $\Omega(X)$, but this correspondence is, in general, not one-to-one nor onto (see [7] for more information on the relation between sublocales and subspaces).

It is well known that every locale has a largest (in fact, unique) Boolean dense sublocale, which coincides with the least dense sublocale [4] — the Booleanization of the locale [1]. This is typically a pointless locale, in the sense that for any Hausdorff space without isolated points it does not contain any points at all.

Moreover, Caramello [2] (cf. also [3]) showed that every topos has a largest dense De Morgan subtopos. By applying it to toposes of sheves over locales, this immediately implies that every locale has a largest dense De Morgan sublocale, where we recall that a locale is said to be *De Morgan* or *extremally disconnected* if the identity

$$(a \wedge b)^* = a^* \vee b^*$$

holds for all $a, b \in L$ (see [5] for more information and other equivalent conditions).

The study of the DeMorganization directly for locales recently started in [6], where a direct proof of its existence was given using nuclei.

In this talk, we will give a direct, simpler, proof of the existence of the DeMorganization in terms of sublocales as concrete subsets, represented as in [7]. This helps understand the topological nature of the DeMorganization. Among others, we will show that, similarly to the Booleanization, the DeMorganization is also a fitted sublocale — i.e. one which occurs as an intersection of open sublocales.

Using these techniques, we will show that for every metric space without isolated points its DeMorganization coincides with its Booleanization (a proof was announced in the abstract of [6], but it was not materialized during the talk).

Time permitting, we will also look at analogues for infinite variants of De Morgan law in locale theory.

References

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