NP-hardness of promise colouring graphs via homotopy

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The so-called algebraic approach to constraint satisfaction problems is well-established and successful example of application of universal algebra in computational complexity. This line of research started with a conjecture of Feder and Vardi [3], that each finite-template CSP is either in P or NP-complete. The algebraic theory of polymorphisms was established by Jeavons et al. [5, 1], and the approach culminated with two independent positive resolutions of the conjecture by Bulatov [2] and by Zhuk [8].

Fixed-template constraint satisfaction problems (CSPs) may be defined in several ways: The goal is to decide whether a given primitive positive formula is satisfiable in a fixed finite structure (called *template*). Alternatively, it is a homomorphism problem for finite relational structures where the target structure is fixed, i.e., we are asking given a structure X whether there is a homomorphism to a fixed structure A. The problem is usually denoted by $\mathrm{CSP}(A)$.

A substantial recent effort has been dedicated to a slight generalisation of CSPs to promise problems. A promise problem consists of two disjoint (but not necessarily complementary) sets of instances: positive and negative. A *promise CSP* is a promise problem whose positive instances are positive instance of $\mathrm{CSP}(A)$ and whose negative instances are negative instances of another $\mathrm{CSP}(B)$; note that in order for these to be disjoint, we have to have a homomorphism from A to B. A prototypical example of a promise CSP is approximate graph colouring: Given a graph G, decide between the case that G is 3-colourable and the case that G is not even 6-colourable. The algebraic methods generalise to the promise setting, but universal algebra is less relevant in resolving the complexity of these problems, and new tools need to be developed to address these problems.

The goal of this talk is to share one of these new tools. I will outline a new application of topology (more precisely, homotopy theory) in assessing the complexity of promise CSPs. Namely, I will talk about hardness of two versions of graph and hypergraph colouring:

- It is NP-complete to decide between graphs that (A) map homomorphically to an odd cycle and those that (B) are not 3-colourable (Krokhin, Opršal, Wrochna, and Živný [6]).
- It is NP-complete to decide between 3-uniform hypergraphs (A) that can be coloured by 3 colours in such a way that each edge has a unique maximum, and (B) those that cannot be coloured by 4 colours in the same way. (Filakovský, Nakajima, Opršal, Tasinato, and Wagner, [4]).

Both proofs are based on topological ideas first used to show lower bounds of the chromatic number of Kneser graphs by Lovász [7] in conjunction with algebraic and categorical tools.

References

[1] Andrei Bulatov, Peter Jeavons, and Andrei Krokhin. Classifying the complexity of constraints using finite algebras. *SIAM Journal on Computing*, 34(3):720–742, 2005. doi:10.1137/S0097539700376676.

[2] Andrei A. Bulatov. A dichotomy theorem for nonuniform CSPs. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 319–330, Berkeley, CA, USA, 2017. IEEE. doi:10.1109/FOCS.2017.37.

[3] Tomás Feder and Moshe Y. Vardi. The computational structure of monotone monadic SNP and constraint satisfaction: A study through datalog and group theory. *SIAM Journal on Computing*, 28(1):57–104, 1998. doi:10.1137/S0097539794266766.

[4] Marek Filakovský, Tamio-Vesa Nakajima, Jakub Opršal, Gianluca Tasinato, and Uli Wagner. Hardness of linearly ordered 4-colouring of 3-colourable 3-uniform hypergraphs, 2023. arXiv:2312.12981, doi:10.48550/arXiv.2312.12981.

[5] Peter Jeavons, David A. Cohen, and Marc Gyssens. Closure properties of constraints. *Journal of the ACM*, 44(4):527–548, 1997. doi:10.1145/263867.263489.

[6] Andrei A. Krokhin, Jakub Opršal, Marcin Wrochna, and Stanislav Živný. Topology and adjunction in promise constraint satisfaction. *SIAM Journal on Computing*, 52(1):38–79, 2023. arXiv:2003.11351, doi:10.1137/20M1378223.

[7] László Lovász. Kneser's conjecture, chromatic number, and homotopy. *J. Combin. Theory Ser. A*, 25(3):319–324, 1978. doi:10.1016/0097-3165(78)90022-5.

[8] Dmitriy Zhuk. A proof of the CSP dichotomy conjecture. *J. ACM*, 67(5):30:1–30:78, 2020. doi:10.1145/3402029.