On Geometric Implications

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It is a well-known fact that although the poset of opens of a topological space is a Heyting algebra, its Heyting implication is not necessarily stable under the inverse image of continuous functions and hence is not a geometric concept. This leaves us wondering if there is any stable family of implications that can be safely called geometric. In this talk, after providing a formalization for geometricity of a family of implications over a category of spaces, we first present a classification for all geometric families over a given subcategory of **Top** satisfying some closure properties and then we show that over the full category **Top**, there is only one geometric family, consisting of trivial implications in a certain sense described below. In the rest of this extended abstract, we will present the formal version of the classification we mentioned above. Let us first start with the abstract notion of implication.

Definition 1. Let $\mathcal{A} = (A, \leq, \wedge, \vee, 1, 0)$ be a bounded distributive lattice. A binary operator $\to$ over $\mathcal{A}$, decreasing in its first argument and increasing in its second is called an *implication* over $\mathcal{A}$ if $a \to a = 1$, for any $a \in \mathcal{A}$ and $(a \to b) \wedge (b \to c) \leq a \to c$, for any $a, b, c \in \mathcal{A}$. An implication is called *weakly boolean* if $a \to b = (a \to 0) \vee b$, for any $a, b \in \mathcal{A}$. If $\to$ is an implication over the lattice of the opens of a space X, denoted by $\mathcal{O}(X)$, then the pair $(X, \to)$ is called a *strong space*. A *strong space map* is a continuous map between spaces such that its inverse image preserves the implication.

Example 2. Over any bounded distributive lattice $\mathcal{A}$, there is a *trivial implication* defined by $a \to_t b = 1$, for any $a, b \in \mathcal{A}$. The Boolean and the Heyting implications are also implications. Notice that the trivial and the boolean implications are weakly boolean.

The second element we must present is the geometricity. Intuitively, geometricity is the stability of a family of implications under the inverse image of a family of continuous functions. Therefore, to formalize this notion, we need to be precise about two ingredients: the continuous maps we use and the family of implications we choose. For the former, it is reasonable to start with a subcategory $\mathcal{S}$ of **Top** to have a relative version of geometricity. For the latter, as any implication must be over a space in this case, a natural formalization of a family of implications is some sort of fibration that to each space X in $\mathcal{S}$ assigns a fiber of strong spaces over X. Having these two ingredients fixed, the geometricity simply means the stability of the fibres under the inverse image of the maps in $\mathcal{S}$. In other words, it states that for any map $f : X \to Y$ in $\mathcal{S}$, the inverse image map f^{-1} must map a fiber over Y into a fiber over X. The following is the formalization of this idea.

Definition 3. Let $\mathcal{S}$ be a (not necessarily full) subcategory of **Top**. A category $\mathcal{C}$ of strong spaces is called *geometric over* $\mathcal{S}$, if the forgetful functor $U : \mathcal{C} \to$ **Top** mapping $\mathcal{C}$ into $\mathcal{S}$, is surjective on the objects of $\mathcal{S}$, and for any object $(Y, \to_Y)$ in $\mathcal{C}$, any object X in $\mathcal{S}$ and any map $f : X \to Y = U(Y, \to_Y)$ in $\mathcal{S}$, there exists an object $(X, \to_X)$ in $\mathcal{C}$ such that f induces a strong space map $f : (X, \to_X) \to (Y, \to_Y)$ in $\mathcal{C}$:

$$
\begin{array}{ccccc}
\mathcal{C} & & (X, \to_X) & \xrightarrow{\ f\ } & (Y, \to_Y) \\
\Big\downarrow U & & \rightsquigarrow & & \rightsquigarrow \\
\mathcal{S} & & X & \xrightarrow[\ f\ ]{} & Y = U(Y, \to_Y)
\end{array}
$$

Note that using the functor U, the category $\mathcal{C}$ is nothing but a way to provide a fiber of strong spaces or equivalently a fiber of implications over any space in $\mathcal{S}$. Then, the conditions simply demand that the fibers and the maps between them are all lying over $\mathcal{S}$ and none of the fibers are empty and the last condition is the geometricity condition we discussed above.

Example 4. For any category $\mathcal{S}$ of spaces, let $\mathcal{S}_t$ be the category of strong spaces $(X, \to_t)$, where X is in $\mathcal{S}$ and $\to_t$ is the trivial implication together with all the maps of $\mathcal{S}$ as the morphisms. It is clear that $\mathcal{S}_t$ is a geometric category over $\mathcal{S}$. To have more examples, recall that a space X is called *indiscrete* if its only opens are $\varnothing$ and X and it is *locally indiscrete* if each $x \in X$ has an indiscrete neighbourhood. Now, if $\mathcal{S}$ only consists of locally indiscrete spaces, then there are three other degenerate geometric categories over $\mathcal{S}$. The first is the category $\mathcal{S}_b$ of strong spaces $(X, \to_b)$, where X is in $\mathcal{S}$ and $\to_b$ is the Boolean implication together with the maps of $\mathcal{S}$ as the morphisms. This category is well-defined, since the locally indiscreteness of X implies the Booleanness of $\mathcal{O}(X)$ and the inverse images always preserve all the Boolean operators. It is easy to see that $\mathcal{S}_b$ is actually geometric over $\mathcal{S}$. The second example is the union of $\mathcal{S}_b$ and $\mathcal{S}_t$ that we denote by $\mathcal{S}_{bt}$. This category is also clearly geometric over $\mathcal{S}$. The third example is $\mathcal{S}_a$, the subcategory of strong spaces $(X, \to)$, where X is in $\mathcal{S}$ and $\to$ is a weakly boolean implication, together with the strong space morphisms that U maps into $\mathcal{S}$. It is not trivial but one can show that $\mathcal{S}_a$ is also geometric over $\mathcal{S}$.

Definition 5. A subcategory $\mathcal{S}$ of **Top** is called *local* if it has at least one non-empty object and it is closed under all embeddings, i.e., for any space X in $\mathcal{S}$ and any embedding $f : Y \to X$, both Y and f belongs to $\mathcal{S}$. A space X is called *full* in $\mathcal{S}$ if it has X as an object and all maps into X as its maps.

The following theorem provides a characterization for all geometric categories over local subcategories of **Top**:

Theorem 6. *Let $\mathcal{S}$ be a local subcategory of* **Top** *with a terminal object:*

(*i*) *If $\mathcal{S}$ has at least one non-locally-indiscrete space, then the only geometric category over $\mathcal{S}$ is $\mathcal{S}_t$.*

(*ii*) *If $\mathcal{S}$ only consists of locally-indiscrete spaces, includes a non-indiscrete space and a full discrete space with two points, then the only geometric categories over $\mathcal{S}$ are the four distinct categories $\mathcal{S}_t$, $\mathcal{S}_b$, $\mathcal{S}_{bt}$ and $\mathcal{S}_a$.*

(*iii*) *If $\mathcal{S}$ only consists of indiscrete spaces, then the only geometric categories over $\mathcal{S}$ are the three distinct categories $\mathcal{S}_t$, $\mathcal{S}_b$, and $\mathcal{S}_{bt}$.*

As a special case, we can see that there is only one geometric category over the whole category **Top**, namely the one with the trivial implications.

Corollary 7. $\mathbf{Top}_t$ *is the only geometric category over* **Top**.

Therefore, one can conclude that there is no non-trivial and fully-geometric notion of implication.