Weil algebras and varieties of rigs

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As explained in [3], some of Grothendieck's algebro-geometric constructions may be abstracted to the context of extensive categories. A category C with finite coproducts is extensive if the canonical functor $\mathsf{C}/X \times \mathsf{C}/Y \to \mathsf{C}/(X+Y)$ is an equivalence for every pair of objects X, Y in C. Extensivity attempts to make explicit a most basic property of (finite) coproducts in categories 'of spaces'. For instance, the category of topological spaces and continuous functions between them is extensive; the category of groups is not. It easily follows that if C is extensive then for any $X \in \mathsf{C}$ the category X/C is extensive [1].

Experience indeed confirms that conceiving an extensive category as a category 'of spaces' is a useful conceptual guide. Essential to the development of Algebraic Geometry is the fact that the opposite of the category of (commutative unital) rings, is extensive.

A category C is coextensive when its opposite category C^{op} is coextensive. In this work we examine the variety of algebras known as *rigs*, denoted Rig, which are commutative semirings with (additive and multiplicative) unit. Of particular interest are those subvarieties $2\mathsf{Rig}$ of (additively) idempotent rigs, as well as the variety iRig of *integral rigs*; those satisfying $1 + x \approx 1$. Such classes play an important role, for instance, in non-classical logics in that these algebras are exactly the (integral) join-semilattice reducts of (pointed) commutative residuated lattices, or $\mathsf{FL_e}$-algebras (respectively, $\mathsf{FL_{ew}}$), semantics for certain extensions of the Full Lambek calculus. Viewed as categories, these classes are coextensive (see [2,4]), and thus admit to the prospect of geometric content.

Let C be a category with a terminal object 1. If X is an object of C, a *point* of X is an arrow $1 \to X$. An object is called *Weil* if it has a unique arrow to the terminal object. At least in the case when the category is a variety of algebras, the terminal object is the free 0-generated algebra. In the case of the variety of rigs, the terminal object is the rig of natural numbers $\mathbb{N}$, while for (non-trivial) subvarieties of 2-rigs the terminal object is always the two element chain $\mathsf{2}$. We note that there is no finite Weil algebra in the in Rig.

An arrow $f\colon X \to Y$ in C is called *constant* if it factors through 1. More generally, an arrow $f\colon X \to Y$ is called a *pseudo-constant* if it coequalizes all the points of X. That is,

$$1 \underset{a}{\overset{b}{\rightrightarrows}} X \xrightarrow{f} D$$

for every pair of points $a, b\colon 1 \to X$, one has $f(a) = f(b)$. Of course, every constant is a pseudo-constant.

Let us write Aff for the opposite of $2\mathsf{Rig}$, and if A is an object in $2\mathsf{Rig}$, let us write A' for the corresponding object in Aff. Trivially, points of A' in Aff are in bijective correspondence with maps $A \to \mathsf{2}$ in $2\mathsf{Rig}$. So, for example, A is a Weil 2-rig iff A' has exactly one point. A map $f\colon A \to B$ is called *pseudo-stant* if for every $g, h\colon B \to \mathsf{2}$ one has $g \circ f = h \circ f$. So, a map is pseudo-stant in the category iR if and only if the corresponding $B' \to A'$ in Aff is a pseudo-constant. Experience with Set suggests that pseudo-constants are constant, but this is too naive. What is sometimes the case in categories of spaces is that the image of a pseudo-constant has exactly one point. This is the content of the following question.

Question 1. Let $\mathcal{V}$ be a variety of rigs. Let $f\colon A \to B$ be such that for every $g, h\colon B \to \mathsf{2}$, $g \circ f = h \circ f\colon A \to \mathsf{2}$. Is it the case that f factors through one Weil algebra in $\mathcal{V}$?

Part of this work is devoted to providing an answer to the above question. In the case for classes of 2-rigs, in particular irigs, we answer this question in the affirmative. This is, in part, a consequence of the following characterization.

Theorem 2. *Let R be any 2-rig. Then the following are equivalent.*

1. *R is a Weil 2-rig.*
2. *R has a unique prime ideal closed under $\leq$.*
3. *R satisfies the following:*

$$\text{For all } x \in R, \exists n \in \mathbb{N}, x^n \leq 0 \quad \text{or} \quad \exists r \in R, 1 \leq rx.$$

where $\leq$ is the partial order defined via $x \leq y$ iff $x + y = y$ in R.

Moreover, the theorem above can be used to establish that the variety of 2-rigs is generated by a its Weil members, in particular this class can be taken to consist of finite algebras of a certain form.

Theorem 3. *For $\mathcal{V}$ taken to be the variety of 2-rigs or the variety of integral rigs, $\mathcal{V}$ is generated by a class of its finite Weil members. Specifically, each finitely generated free-algebra is a subdirect product of finite Weil algebras in $\mathcal{V}$ [satisfying a stronger version of item (3)].*

References

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