Evolution systems: amalgamation, absorption, and termination

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We introduce the concept of an abstract evolution system which provides a convenient framework for studying generic mathematical structures. The talk is based on a joint work with W. Kubiś [1].

Definition. An ***evolution system*** is a structure of the form $\mathcal{E} = \langle \mathfrak{V}, \mathcal{T}, \Theta \rangle$, where $\mathfrak{V}$ is a category, Θ is a fixed $\mathfrak{V}$-object (called the ***origin***), and $\mathcal{T}$ is a class of $\mathfrak{V}$-arrows (its elements are called ***transitions***) satisfying

$$\forall_{X \in \mathrm{Obj}(\mathfrak{V})} \quad \mathrm{id}_X \in \mathcal{T} \quad \text{and} \quad \forall_{t \in \mathcal{T}} \forall_{h \in \mathrm{Iso}(\mathfrak{V})} \quad h \circ t \in \mathcal{T}.$$

Main focus of this concept resolves around ***evolutions***, namely sequences of the form

$$\Theta \to A_0 \to A_1 \to \cdots \to A_n \to \cdots$$

where each of the arrows above is a transition. With every evolution $\vec{a}$ we will associate its colimit $\lim \vec{a} = A_\infty$. A finite composition of transitions will be called a ***path*** An object X is ***finite*** if there exists a path from the origin to X. A ***σ-path*** from A to B is the colimiting arrow in a sequence of transitions of the form $A = A_0 \to A_1 \to \cdots$ with colimit B.

Example. Let $\mathfrak{V}$ be the category whose objects are first-order structures of a fixed language, e.g., graphs, ordered sets, (semi-)groups, etc. We turn it into a natural evolution system $\mathcal{E}$. Namely, a transition from X to Y will be an embedding $t\colon X \to Y$ such that either $Y = t[X]$ or Y is generated by $t[X] \cup \{v\}$ for some $v \in Y \setminus t[X]$. Finally, we need to set the origin Θ, which typically is the trivial structure or any of the simplest finite structures of our choice.

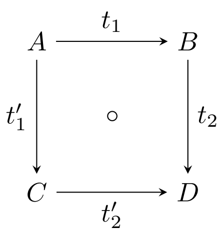
Later on, we consider the following properties of both the whole system and of particular evolution of interest. Given a $\mathfrak{V}$-object X, we denote by $\mathcal{T}(X) = \{f \in \mathcal{T}\colon \mathrm{dom}(f) = X\}$, the class of all transitions with domain X. Arrows f, g are said to be ***left-isomorphic*** if $g = h \circ f$ for some isomorphism h. We say that $\mathcal{E}$ is ***locally countable*** if $\mathcal{T}(X)$ has countably many left-isomorphism classes for every finite object X.

We say that an evolution system $\mathcal{E}$ has ***transition amalgamation property*** (TAP) if for every pair of transitions $t_1\colon A \to B$ and $t_1'\colon A \to C$, where A is a finite object, there exist transitions $t_2\colon B \to D$ and $t_2'\colon C \to D$ such that $t_2 \circ t_1 = t_2' \circ t_1'$.

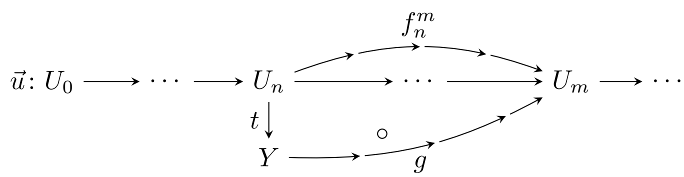
Similarly $\mathcal{E}$ has the ***amalgamation property*** (AP) if for every two paths f, g with $\mathrm{dom}(f) = \mathrm{dom}(g)$ finite, there exist paths f', g' such that $f' \circ f = g' \circ g$. Clearly TAP $\implies$ AP, while the converse is false.

*Research of both authors supported by GACR grant EXPRO 20-31529X

Let $\vec{u}$ be an evolution. We say that $\vec{u}$ has the ***absorption property (for transitions)*** if for every $n \in \omega$, for every path (transition) $t\colon A_n \to Y$ there are $m \geq n$ and a path $g\colon Y \to A_m$ such that $g \circ t = f_n^m$.



TAP



Absorption

Theorem. Assume $\mathcal{E}$ is a locally countable evolution system with the TAP. Then there exists an evolution $\vec{u}$ with the absorption property. Moreover, let U be the colimit of $\vec{u}$. Then

(1) Every finite object admits a σ-path into U.

(2) For every finite object A, for every two σ-paths f_0, f_1 from A to U there exists an automorphism $h\colon U \to U$ such that $f_1 = h \circ f_0$.

Example. Consider a category of graphs; as the origin let us take a single vertex, and a transition is adding one vertex connected with some of the already existing ones. We obtain different evolution systems depending on how this new edge is connected: to all vertices, to none of them, at random, to 10% of existing vertices, to (at most) k of them and so on. In the talk we will discuss how it influences the colimit of an evolution with the absorption property.

We end with a brief discussion on ***terminating*** evolution systems, namely systems in which every evolution is eventually trivial, that is, from some point on all transitions are isomorphisms. A finite object N is ***normalized*** if every transition from N is an isomorphism. An evolution system $\mathcal{E}$ is ***regular*** if $t \circ h \in \mathcal{T}$ is a transition whenever $t \in \mathcal{T}$ and h is an isomorphism. An evolution system $\mathcal{E}$ is ***locally confluent*** if for every two transitions f, g with $\mathrm{dom}(f) = \mathrm{dom}(g)$ finite, there exist paths f', g' satisfying $f' \circ f = g' \circ g$.

Theorem. Every regular locally confluent terminating evolution system has the amalgamation property.

Theorem. Let $\mathcal{E}$ be a regular locally confluent terminating evolution system. Then there exists a unique, up to isomorphism, normalized object U. Furthermore

(1) Every finite object admits a path into U.

(2) For every finite object A, for every two paths f_0, f_1 from A to U there exists an automorphism $h\colon U \to U$ such that $f_1 = h \circ f_0$.

References

[1] P. Radecka W. Kubiś. Evolution systems: amalgamation, absorption, and termination . https://arxiv.org/abs/2109.12600, 2023.