Enriched and Homotopical Coalgebra

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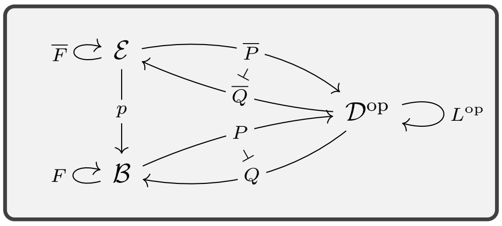
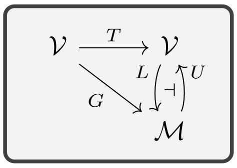
Coalgebra has emerged from the desire to find an abstraction of the behaviour of computational models [Rut00]. It starts with the insight that behaviour of many systems arises by repeated observation of a morphism $c\colon X \to FX$, where the kind of observations that can be made are determined by a functor $F\colon \mathcal{C} \to \mathcal{C}$ on a category $\mathcal{C}$. The idea is that FX is the space of possible observations on X that the coalgebra c yields. Instances of this view are transition systems, concurrent systems, probabilistic and timed systems, coinductive proofs, and various systems with topological structure, such as topological models of modal logic, dynamical systems and hybrid systems. A coalgebra c gives rise to behaviour in form of a sequence $X \xrightarrow{c} FX \xrightarrow{Fc} F(FX) \xrightarrow{F(Fc)} \cdots$ that recursively expands the observations. If this sequence approaches a limit, then this limit can be interpreted as total view on the behaviour of c [Bar93].

In this talk, I wish to present developments of enriched coalgebra in two main directions. The first direction is a theory of enriched categories and fibrations of coalgebras. Enriched category theory allows us to apply coalgebra to a wide variety of areas, which are not captured by categories with sets of morphisms. For instance, we can instead consider coalgebras in metric spaces, in order-enriched categories [BKPV11, BK11], topological or simplicial categories etc. In this direction, I aim to first present a few basic results and examples on enrichment, weighted (co)limits and (co)tensors for coalgebras. Then we turn to coalgebraic modal logic [CKP$^+$11, Mos99], which allows us to make partial observations on the recursive sequence mentioned above. Over plain categories, various correspondence results between bisimilarity and logical equivalence have been obtained [Kli07, Pat03, Sch08], and they have been extend to coalgebras in enriched categories [BD13, Wil12, Wil13]. Recently, it was shown how results in coalgebraic modal logic can be extended to other predicates by modelling the target predicate as a fibration map $(F, \overline{F})$ on a fibration $p\colon \mathcal{E} \to \mathcal{B}$, the modal logic as initial algebra for a functor L on a suitable category $\mathcal{D}$ of algebras, and the relation between the two by a pair of dual adjunction as in the diagram on the right [KR21]. Whenever the two adjunctions are related by distributive laws and $\mathcal{B}$ comes with a factorisation system, we can general obtain soundness and completeness results. My goal is to present an enriched version of this approach to enriched coalgebraic modal logic, where the fibration etc. are suitably enriched.



The second direction of development concerns enriched Kleisli categories. The Kleisli category of a monad is a well-known model for programs with computational effects. If the Kleisli category is enriched, then this enrichment provides an account of other computational features, such as recursion via CPO-enrichment. I will show how to obtain an $\mathcal{M}$-enrichment for the Kleisli category of a monad T on a category $\mathcal{V}$, even though $\mathcal{V}$ may not be $\mathcal{M}$-enriched, if the monad factor through the right-adjoint U of a suitable adjunction as in the diagram on the left. This result covers examples like order- and CPO-enrichment in case of the powerset and distribution monad that are typical in program semantics. We will also look at topological enrichment, which is the base of a homotopy theory for coalgebra, and can be used in topological models of modal logic [GT22, KKV04, Bal03, VdB22] and hybrid systems [NB18, Nev17].



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