A Categorical Characterization of the Low-Complexity Functions

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A function is primitive recursive iff it is representable by a map on the parameterized initial $F_{\mathbf{N}}$-algebra of any cartesian category (if it exists), where $F_{\mathbf{N}}(X) = 1 + X$. In this talk, following the philosophy of predicativism, we weaken the definition of a parameterized initial F-algebra to introduce a new notion called a *predicative F-scheme*, for any endofunctor F. Then, we show that the predicative $F_{\mathbf{N}}$-scheme (resp. predicative $F_{\mathbf{W}}$-scheme, where $F_{\mathbf{W}}(X) = 1 + X + X$) naturally captures the class of all linear space (resp. polynomial time) computable functions as its all and only representable functions. In the rest of this extended abstract, we will present the definitions of predicative F-schemes and representability to make the above points more formal.

First, we need to recall some basic definitions. Let $\mathcal{C}$ be a cartesian category (i.e., with all finite products), $F : \mathcal{C} \to \mathcal{C}$ be a functor and X be an object in $\mathcal{C}$. By an *F-algebra in $\mathcal{C}$ with parameters in X*, we mean the tuple $\mathbf{A} = (X, A, a)$, where $a : X \times F(A) \to A$ is a map in $\mathcal{C}$. The object A is called the *carrier* of $\mathbf{A}$ and is denoted by $|\mathbf{A}|$. When $X = 1$, an F-algebra with parameters in X is simply called an F-algebra. For any two F-algebras $\mathbf{A} = (X, A, a)$ and $\mathbf{B} = (X, B, b)$ in $\mathcal{C}$ with parameters in X, by an *F-homomorphism*, we mean a $\mathcal{C}$-map $f : A \to B$ such that the following diagram commutes:

$$\begin{array}{ccc} X \times F(A) & \xrightarrow{\;a\;} & A \\ {\scriptstyle id_X \times F(f)} \big\downarrow & & \big\downarrow {\scriptstyle f} \\ X \times F(B) & \xrightarrow[\;b\;]{} & B \end{array}$$

It is clear that F-algebras in $\mathcal{C}$ with parameters in X together with F-homomorphisms form a category denoted by $\mathbf{Alg}^F_X(\mathcal{C})$. Moreover, the assignment $| - | : \mathbf{Alg}^F_X(\mathcal{C}) \to \mathcal{C}$ mapping an F-algebra with parameters in X to its carrier and an F-homomorphism to itself is a functor. Also, note that any $g : X \to Y$ in $\mathcal{C}$ induces a canonical functor $g^* : \mathbf{Alg}^F_Y(\mathcal{C}) \to \mathbf{Alg}^F_X(\mathcal{C})$.

Definition 1. Let $\mathcal{E}$ be a cartesian category, $\mathcal{D}$ be its (not necessarily full) cartesian subcategory, $i : \mathcal{D} \to \mathcal{E}$ be the inclusion functor preserving all finite products, and $F : \mathcal{E} \to \mathcal{E}$ be a functor whose restriction to $\mathcal{D}$ lands in $\mathcal{D}$ itself. An object I in $\mathcal{E}$ is called the *F-scheme of $\mathcal{D}$ in $\mathcal{E}$*, if for any $X \in \mathcal{D}$, the object $X \times I$ is the limit of the diagram $i| - | : \mathbf{Alg}^F_X(\mathcal{D}) \to \mathcal{E}$ via the cone $\langle r_{X,\mathbf{A}} \rangle_{\mathbf{A} \in \mathbf{Alg}^F_X(\mathcal{D})}$ and for any $\mathcal{D}$-map $f : X \to Y$ and any F-algebra $\mathbf{A}$ in $\mathcal{D}$ with parameters in Y, the following diagram commutes:

$$\begin{array}{ccc} I \times X & & \\ {\scriptstyle id_I \times f} \big\downarrow & \searrow^{r_{X, f^* \mathbf{A}}} & \\ I \times Y & \xrightarrow[\;r_{Y,\mathbf{A}}\;]{} & |\mathbf{A}| \end{array}$$

The F-scheme of $\mathcal{D}$ in $\mathcal{E}$ is meant to formalize the common *scheme* of all F-algebras of $\mathcal{D}$ (with parameters) inside the possibly greater category $\mathcal{E}$. Using the universality of the limit, one can easily show that there is a canonical F-algebra structure $a_I : F(I) \to I$ on I, whose composition with the projection provides an F-algebra structure on $I \times X$ with parameters in X. It is also easy to see that this algebraic structure makes all $r_{X,\mathbf{A}}$'s into F-homomorphisms.

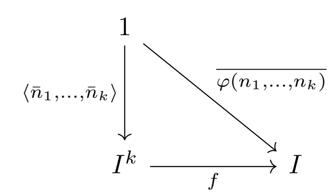
In the special case when $\mathcal{E} = \mathcal{D}$, the F-scheme of $\mathcal{D}$ in $\mathcal{D}$ is nothing but the initial F-algebra:

Theorem 2. *Let $\mathcal{D}$ be a finitely complete category. If I is the F-scheme of $\mathcal{D}$ in $\mathcal{D}$, then the F-algebra $a_I : F(I) \to I$ is the parameterized initial F-algebra in $\mathcal{D}$. Conversely, if $\mathbf{A}$ is the parameterized initial F-algebra in $\mathcal{D}$, then the object $|\mathbf{A}|$ together with its unique F-homomorphisms into the F-algebras of $\mathcal{D}$ (with parameters) is the F-scheme of $\mathcal{D}$ in $\mathcal{D}$.*

In the general situation when $\mathcal{E}$ is different from $\mathcal{D}$, we need to add an additional property, called the *approximability*, to gain a more well-behaved F-scheme. Roughly speaking, although the limit of the diagram $i|-| : \mathbf{Alg}^F_X(\mathcal{D}) \to \mathcal{E}$ may not belong to $\mathcal{D}$, we want it to be approximable by the objects inside the smaller category $\mathcal{D}$. More formally, the category $Alg^F_X(\mathcal{D})$ is called approximable iff there is a directed family $\{\mathcal{S}_j\}_{j \in J}$ of classes of morphisms of $\mathcal{D}$ (not necessarily closed under composition) such that it covers the whole $Morph(\mathcal{D})$ and the restriction of $Alg^F_X(\mathcal{D})$ to $\mathcal{S}_j$ has an initial element, for any $j \in J$. Unfortunately, the fully formal definition of approximability is beyond the scope of this short abstract. The reason is some subtleties in the definitions of the restriction, the initial element and the compatibility in the parameter object X, all because the $\mathcal{S}_j$'s are not necessarily closed under the composition.

Having approximability defined, the F-scheme of $\mathcal{D}$ in $\mathcal{E}$ is called *predicative* if $Alg^F_X(\mathcal{D})$ is approximable.

Now, we turn to the representability. Let $\mathbf{N} = (\mathbb{N}, s, 0)$ and $\mathbf{W} = (\mathbb{W}, s_0, s_1, \epsilon)$ be the usual algebras of natural numbers and binary strings, where $s(n) = n+1$, $s_0(w) = w0$, $s_1(w) = w1$ and ϵ is the empty string. In the rest, let us assume that $\mathcal{D}$ and $\mathcal{E}$ are both cartesian and cocartesian categories, $i : \mathcal{D} \to \mathcal{E}$ preserves these structures and $F_{\mathbf{N}} : \mathcal{E} \to \mathcal{E}$ and $F_{\mathbf{W}} : \mathcal{E} \to \mathcal{E}$ be the functors defined by $F_{\mathbf{N}}(X) = 1 + X$ and $F_{\mathbf{W}}(X) = 1 + X + X$. It is possible to represent any element $n \in \mathbb{N}$ (resp. $w \in \mathbb{W}$) as a map in $Hom_{\mathcal{E}}(1, I)$, if I is the $F_{\mathbf{N}}$-scheme (resp. $F_{\mathbf{W}}$-scheme) of $\mathcal{D}$ in $\mathcal{E}$. Denote this canonical representation by $\bar{n}$ (resp. $\bar{w}$). Similarly, we say that an $\mathcal{E}$-map $f : I^k \to I$ represents a function $\varphi : \mathbb{N}^k \to \mathbb{N}$ if the following commutes:



for any $(n_1, \ldots, n_k) \in \mathbb{N}^k$. One can have a similar definition replacing $\mathbb{N}$ by $\mathbb{W}$. Now, we are finally ready to present our main result:

Theorem 3. *(i) A function $\varphi : \mathbb{N}^k \to \mathbb{N}$ is linear space computable iff it is representable as a map on the predicative $F_{\mathbf{N}}$-scheme of $\mathcal{D}$ in $\mathcal{E}$, for any $\mathcal{D}$ and $\mathcal{E}$.*

(ii) A function $\varphi : \mathbb{W}^k \to \mathbb{W}$ is polynomial time computable iff it is representable as a map on the predicative $F_{\mathbf{W}}$-scheme of $\mathcal{D}$ in $\mathcal{E}$, for any $\mathcal{D}$ and $\mathcal{E}$.