# Gödel–Dummett CTL


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Gödel–Dummett logic is a well-known and extensively studied multivalued logic [5]. It is both a superintuitionistic logic and a t-norm fuzzy logic. Computation tree logic (CTL) [4] is a branching-time temporal logic that is a relative of linear temporal logic (LTL) (both are fragments of CTL*). Both LTL and CTL were designed and have been used very successfully for formal verification.

Although nonclassical variants of modal and temporal logics often compare unfavourably to their classical counterparts in terms of logical and computational properties [6, 3], recent investigations have shown that Gödel–Dummett logic pairs well with linear temporal logic. Indeed the variant of LTL whose modality-free fragment is Gödel–Dummett logic is not only decidable, but has an optimal PSPACE complexity [2], and a finite Hilbert-style calculus has been given for Gödel–Dummett LTL enriched with the "coimplication" connective of bi-intuitionistic logic [1].

In this talk we report on similar investigations into a Gödel–Dummett CTL and show that it too is decidable.

Fix a countably infinite set $\mathbb{P}$ of propositional variables. Then the **bi-intuitionistic CTL language** $\mathcal{L}$ is the language defined by the grammar (in Backus–Naur form):

$$\varphi \coloneqq p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \varphi \prec \varphi \mid \exists \mathsf{X}\varphi \mid \forall \mathsf{X}\varphi \mid \exists \mathsf{G}\varphi \mid \forall \mathsf{F}\varphi \mid \exists(\varphi \,\mathsf{U}\, \varphi) \mid \forall(\varphi \,\mathsf{R}\, \varphi),$$

where $p \in \mathbb{P}$. Here, an $\exists$ is read as 'there exists a path (from this state)', a $\forall$ as 'for all paths', $\mathsf{X}$ is as 'ne**x**t', $\mathsf{G}$ as '**g**oing (to always be)', $\mathsf{F}$ as '**f**uture', $\mathsf{U}$ as '**u**ntil' and $\mathsf{R}$ as '**r**eleased by'. The connective $\prec$ is *co-implication* and represents the operator that is dual to implication [7]. We can also define the following abbreviations:

- $\top$ abbreviates $p \to p$, and $\bot$ abbreviates $p \prec p$, for some fixed, but unspecified, $p \in \mathbb{P}$;
- $\neg\varphi$ abbreviates $\varphi \to \bot$;
- $\varphi \leftrightarrow \psi$ abbreviates $(\varphi \to \psi) \wedge (\psi \to \varphi)$ (not the formula $(\varphi \to \psi) \wedge (\varphi \prec \psi)$);
- $\forall\mathsf{G}\varphi$ abbreviates $\forall(\varphi \,\mathsf{R}\, \bot)$ and $\exists\mathsf{F}\varphi$ abbreviates $\exists(\top \,\mathsf{U}\, \varphi)$;
- $\forall(\varphi \,\mathsf{U}\, \psi)$ abbreviates $\forall(\varphi \,\mathsf{R}\, \psi) \wedge \forall\mathsf{F}\psi$ and $\exists(\varphi \,\mathsf{R}\, \psi)$ abbreviates $\exists(\varphi \,\mathsf{U}\, \psi) \vee \exists\mathsf{G}\psi$;

We define the Gödel–Dummett CTL logic using two natural semantics (the details of which we do not give here): first a *real-valued semantics*, where statements have a degree of truth in the real unit interval and second a *bi-relational semantics*.

We define:

- the logic $\mathsf{GCTL}_{\mathbb{R}}$ to be the set of $\mathcal{L}$-formulas that are valid with respect to the real-valued semantics;
- the logic $\mathsf{GCTL}_{\mathrm{rel}}$ to be the set of $\mathcal{L}$-formulas that are valid with respect to the bi-relational semantics.

However, any formula falsifiable on a real-valued model is falsifiable on a bi-relational model.

**Proposition 1.** $\mathsf{GCTL}_{\mathrm{rel}} \subseteq \mathsf{GCTL}_{\mathbb{R}}$.

For $\mathsf{GCTL}_{\mathrm{rel}}$, we use a variant of the technical notion of a pseudo-model, as introduced in [4], and adapted here for CTL. We show that every bi-relationally falsifiable statement is falsifiable on a finite pseudo-model, and vice versa. This directly yields an algorithm for deciding if a statement is valid or not.

**Theorem 2.** *The logic* $\mathsf{GCTL}_{\mathrm{rel}}$ *is decidable.*

# References


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