Local finiteness in varieties of MS4-algebras

Guram Bezhanishvili[1] and Chase Meadors[2*]

[1] New Mexico State University, Las Cruces, NM
guram@nmsu.edu
[2] University of Colorado, Boulder, CO
chase.meadors@colorado.edu

An S4-algebra (or closure algebra) is a pair $(B, \Diamond)$ where B is a Boolean algebra and $\Diamond$ is a unary S4-operator on B (a closure operator). S4-algebras provide semantics for the well-known modal logic S4 ([9, 3]). A classical result of Segerberg and Maksimova [10, 7] gives a criterion characterizing when a variety of S4-algebras is locally finite. Via Jónsson–Tarski duality, the variety $\mathbf{S4}$ of S4-algebras is dually equivalent to a category of descriptive frames (X, R), where X is a Stone space and R is a continuous quasi-order on X. It is then meaningful to speak of the *depth* of an S4-algebra, meaning the longest length of a proper R-chain in its dual frame. We say a variety $\mathbf{V} \subseteq \mathbf{S4}$ has depth $\leq n$ if the depth of each algebra from $\mathbf{V}$ is at most n. Considering the well-known family of formulas (see, e.g., [3, p. 81])

$$P_1 = \Diamond\Box q_1 \to \Box q_1 \qquad P_n = \Diamond(\Box q_n \wedge \neg P_{n-1}) \to \Box q_n$$

we have, for a variety $\mathbf{V} \subseteq \mathbf{S4}$,

1. $\mathbf{V} \models P_n$ iff the depth of $\mathbf{V}$ is $\leq n$ (see, e.g., [3, Prop. 3.44])
2. $\mathbf{V}$ is locally finite iff $\mathbf{V} \models P_n$ for some n (Segerberg–Maksimova).

Thus the locally finite varieties of S4-algebras are precisely those of finite depth. In addition, there is a least subvariety of S4 of infinite depth: the variety $\mathbf{Grz.3}$ generated by the algebra whose dual space is an infinite descending chain. As a consequence, we may effectively decide if a given variety $\mathbf{V} \subseteq \mathbf{S4}$ is locally finite by determining whether $\mathbf{V} \supseteq \mathbf{Grz.3}$.

An MS4-algebra is a tuple $(B, \Diamond, \exists)$ where $\Diamond$ is an S4-operator and $\exists$ is an S5-operator. Their dual frames are tuples (X, R, E) where X is a Stone space, R is a continuous quasiorder on X, and E is a continuous equivalence relation on X. In the same manner as S4-algebras, we may speak of the depth of an S4-algebra to refer to the longest length of an R-chain in its dual frame. MS4-algebras provide semantics for the modal logic MS4, which may be understood as axiomatizing the one-variable fragment of predicate S4 (QS4); see [4]. In light of this, it is natural to investigate to what extent the Segerberg–Maksimova theorem generalizes to this setting. We give an overview of several results in this direction that are explored in [2].

In the way of positive results, we identify the largest semisimple subvariety of $\mathbf{MS4}$, denoted $\mathbf{MS4_S}$, which contains two well-known subvarieties corresponding to $\mathsf{S4}_u$ (S4 extended with the universal modality) and $\mathsf{S5}^2$ (the *product* of S5 with itself, also known as the variety of diagonal-free cylindric algebras of dimension two, see e.g. [5]). We demonstrate that a direct generalization of the Segerberg–Maksimova theorem holds for a family of varieties containing $\mathbf{S4}_u$.

On the other hand, it was known (see, e.g., [6]) that the variety $\mathbf{S5}^2$, which is precisely the variety of MS4-algebras of depth-1, is not locally finite; hence the Segerberg–Maksimova theorem does not generalize directly to $\mathbf{MS4}$. We demonstrate that, in fact, characterizing local finiteness in $\mathbf{MS4}$ is at least as hard as the corresponding problem for $\mathbf{S5}_2$, which remains

*Speaker

wide-open. Here $\mathbf{S5}_2$ corresponds to the *fusion* $\mathsf{S5} * \mathsf{S5}$ – the bimodal logic of two (unrelated) $\mathsf{S5}$ modalities (see, e.g., [5]). We establish this by giving a translation T from subvarieties of $\mathbf{S5}_2$ to subvarieties of $\mathbf{MS4_S} + P_2$ that preserves and reflects local finiteness (i.e., $\mathbf{V}$ is locally finite iff $T(\mathbf{V})$ is locally finite). So already in semisimple subvarieties of depth-2, characterizing local finiteness is difficult.

Finally, we discuss another notable subvariety of **MS4**, denoted $\mathbf{M^+S4}$. Casari's predicate formula

$$\mathsf{Cas} := \forall x((P(x) \to \forall y P(y)) \to \forall y P(y)) \to \forall x P(x)$$

is well-known in the study of intermediate predicate logic (see, e.g., [8]). In [1] it is shown that the monadic version of Casari's formula is necessary to obtain a faithful provability interpretation of monadic intuitionistic logic. $\mathbf{M^+S4}$ is the subvariety of **MS4** obtained by asserting the Gödel translation of this formula, which is then natural to study. Preliminary results indicate that the variety $\mathbf{M^+S4}$ has a much more manageable characterization of local finiteness.

References

[1] G. Bezhanishvili, K. Brantley, and J. Ilin. Monadic intuitionistic and modal logics admitting provability interpretations. *J. Symb. Log.*, 88(1):427–467, 2023.

[2] Guram Bezhanishvili and Chase Meadors. Local finiteness in varieties of MS4-algebras. arXiv:1410.5093.

[3] A. Chagrov and M. Zakharyaschev. *Modal logic*, volume 35 of *Oxford Logic Guides*. Oxford University Press, 1997.

[4] G. Fischer-Servi. On modal logic with an intuitionistic base. *Studia Logica*, 36(3):141–149, 1977.

[5] D. M. Gabbay, A. Kurucz, F. Wolter, and M. Zakharyaschev. *Many-dimensional modal logics: theory and applications*. Elsevier North Holland, 2003.

[6] L. Henkin, J. D. Monk, and A. Tarski. *Cylindric algebras. Part I*, volume 64 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 1985. Reprint of the 1971 original.

[7] L. L. Maksimova. Modal logics of finite layers. *Algebra and Logic*, 14(3):188–197, 1975.

[8] Hiroakira Ono. Some problems in intermediate predicate logics. *Rep. Math. Logic*, (21):55–67, 1987.

[9] H. Rasiowa and R. Sikorski. *The mathematics of metamathematics*. PWN—Polish Scientific Publishers, Warsaw, third edition edition, 1970. Monografie Matematyczne, Tom 41.

[10] K. Segerberg. *An essay in classical modal logic*, volume 13 of *Filosofiska Studier*. Uppsala University, 1971.