On semidirect products of biresiduation algebras

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Biresiduation algebras or pseudo-BCK-algebras are the $\{/, \backslash, 1\}$-subreducts of integral residuated po-monoids (or lattices). We will discuss semidirect products of biresiduation algebras, with emphasis on divisible biresiduation algebras.

To begin with, we recall that a *biresiduation algebra* [8] or a *pseudo-BCK-algebra* [3] is an algebra $(A, /, \backslash, 1)$ satisfying the equations $(y/x)\backslash((z/y)\backslash(z/x)) = 1$, $((x\backslash z)/(y\backslash z))/(x\backslash y) = 1$, $1\backslash x = x$, $x/1 = x$, $x\backslash 1 = 1$, $1/x = 1$, and the quasi-equation $(x\backslash y = 1 \ \&\ y\backslash x = 1) \Rightarrow x = y$. Following [9], we call a biresiduation algebra *divisible* if it satisfies the equations $(x\backslash y)\backslash(x\backslash z) = (y\backslash x)\backslash(y\backslash z)$ and $(z/x)/(y/x) = (z/y)/(x/y)$ (which in case of integral residuated po-monoids are equivalent to the divisibility law). By a *closure endomorphism* we mean an endomorphism that is also a closure operator.

Given two nontrivial biresiduation algebras C, D and an action ρ of C on D we define the *semidirect product* $C \ltimes_\rho D$ to be $\{(a, x) \in C \times D\colon \rho(a, x) = x\}$ with $(a, x)\backslash(b, y) = (a\backslash b, x\backslash\rho(a, y))$ and $(b, y)/(a, x) = (b/a, \rho(a, y)/x)$. If the action ρ satisfies certain conditions resembling divisibility and the maps $\rho(a, -)$ are closure endomorphisms of D, then $C \ltimes_\rho D$ is a biresiduation algebra (with a closure endomorphism), and $C \ltimes_\rho D$ is divisible if and only if both C and D are divisible.

This construction is a quite straightforward generalization of *symmetric semidirect products* of the so-called CKL-algebras [6] (which are equivalent divisible BCK-algebras or HBCK-algebras [1]) as well as of *quasidirect products* of Hilbert algebras [2]. In fact, similarly to [4], it goes back to the construction of implicative semilattices from triples consisting of a boolean algebra, an implicative semilattice and an *admissible function* [5].

If A is a divisible biresiduation algebra with a fixed closure endomorphism δ, then $C = \delta(A)$ is a subalgebra of A, $D = \delta^{-1}(1)$ is a filter of A (hence a biresiduation algebra) and, for every $a \in C$, the map $\rho(a, -) = a\backslash -$ is a closure endomorphism of D. Thus we can construct the semidirect product $C \ltimes_\rho D$. Though A is in general smaller than $C \ltimes_\rho D$, the two algebras determine essentially the same triples. In some particular cases, $C \ltimes_\rho D$ is isomorphic to A. For example, this happens when A is a BL-algebra and the fixed closure endomorphism δ is just the double negation (this generalizes the results of [4]).

For divisible biresiduation algebras, we have an adjunction between the category of algebras with closure endomorphisms and the category of "modules"/triples. Specifically, (i) let $\mathcal{A}$ be the category of divisible biresiduation algebras with fixed closure endomorphisms, i.e., algebras (A, δ), with morphisms = homomorphisms, and (ii) let $\mathcal{M}$ be the category of "modules" D over C, i.e., triples (C, D, ρ) where C, D are divisible biresiduation algebras and ρ an action of C on D, with morphisms from (C, D, ρ) to (C_1, D_1, ρ_1) defined as pairs of homomorphisms $f\colon C \to C_1$, $g\colon D \to D_1$ such that $g(\rho(a, x)) = \rho_1(f(a), g(x))$ for all $a \in C$ and $x \in D$. Then, using the assignments "algebra $(A, \delta) \mapsto$ triple (C, D, ρ)" and "triple $(C, D, \rho) \mapsto$ semidirect product $C \ltimes_\rho D$" described above, we define adjoint functors $F\colon \mathcal{A} \to \mathcal{M}$ and $G\colon \mathcal{M} \to \mathcal{A}$, with $F \dashv G$.

We will also discuss the role of n-potent elements and characterize the so-called *quasi-decompositions* (in the sense of [7] or [2]) corresponding to closure endomorphisms of divisible biresiduation algebras.

References

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