Monoidal aspects of cocomplete $\mathcal{V}$-enriched categories

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Fifty years ago, Lawvere's observation that ordered sets and metric spaces can be seen as enriched categories over a quantale opened a wide path for the quantitative theory of domains, using and generalising ideas from category theory, algebra, logic and and topology. Among the most beneficial and pleasing properties of such quantale-enriched categories are undoubtedly (co)completeness with respect to certain classes of (co)limits and commutation of such limits and colimits (with prominent examples featuring the ordered case, like continuous or completely distributive lattices).

The present talk considers the category $\mathcal{V}\text{-}\mathsf{Sup}$ of cocomplete enriched categories over a commutative quantale $\mathcal{V}$ – the $\mathcal{V}$-valued analogue of complete sup-lattices. The arrows are cocontinuous $\mathcal{V}$-functors. Cocomplete $\mathcal{V}$-categories are the Eilenberg-Moore algebras for the free cocompletion monad $\mathbb{D}$ on $\mathcal{V}$-categories. $\mathcal{V}\text{-}\mathsf{Sup}$ is *symmetric monoidal closed*. The corresponding tensor product $\otimes_{\mathcal{V}\text{-}\mathsf{Sup}}$ arises naturally, using that $\mathbb{D}$ is a KZ-monad (hence commutative) [6, 8], and it classifies bimorphisms [1, 7]. The internal hom is $\mathcal{V}\text{-}\mathsf{Sup}(A, B)$. Besides being monoidal closed, $\mathcal{V}\text{-}\mathsf{Sup}$ is also $*$-*autonomous*, with dualizing object $\mathcal{V}^{\mathsf{op}}$ [3]. Consequently, the tensor product $A \otimes_{\mathcal{V}\text{-}\mathsf{Sup}} B$ of two cocomplete $\mathcal{V}$-categories A and B can alternatively be characterised as $\mathcal{V}\text{-}\mathsf{Sup}(A, B^{\mathsf{op}})^{\mathsf{op}}$. In case of complete sup-lattices, this is precisely the set of Galois maps between them [10]. An alternative description of the tensor product of two complete sup-lattices is obtained as the set of all down-sets in their cartesian product that are join-closed in either coordinate [13]. Taking advantage of the 2-categorical setting, we can extend this description to (co)complete quantale-enriched categories.

Proposition 1. *Let A and B be two cocomplete $\mathcal{V}$-categories. Then the coreflexive inverter in $\mathcal{V}\text{-}\mathsf{Sup}$ of the 2-cell (inequality) $\mathbb{D}(\mathsf{y}_A \otimes \mathsf{y}_B) \Rightarrow \mathbb{D}_\forall(\mathsf{y}_A \otimes \mathsf{y}_B)$ is $A \otimes_{\mathcal{V}\text{-}\mathsf{Sup}} B$:*

$$A \otimes_{\mathcal{V}\text{-}\mathsf{Sup}} B \longrightarrow \mathbb{D}(A \otimes B) \underset{\mathbb{D}_\forall(\mathsf{y}_A \otimes \mathsf{y}_B)}{\overset{\mathbb{D}(\mathsf{y}_A \otimes \mathsf{y}_B)}{\overset{\longrightarrow}{\Downarrow \atop \longrightarrow}}} \mathbb{D}(\mathbb{D}(A) \otimes \mathbb{D}(B))$$

In the above, $\mathsf{y}_A : A \to \mathbb{D}(A)$ and $\mathsf{y}_B : B \to \mathbb{D}(B)$ denote the Yoneda embeddings, and $\mathbb{D}_\forall(\mathsf{y}_A \otimes \mathsf{y}_B)$ is the right adjoint to the right adjoint to $\mathbb{D}(\mathsf{y}_A \otimes \mathsf{y}_B)$.

Nuclearity (in modern parlance *dualisability*) originally arose in Operator Theory, in order to mimic finite dimensionality behaviour (for objects) and matrix calculus (for arrows) [4]. It was subsequently observed that nuclearity was in fact a categorical concept, and that it could be defined in the more general context of (symmetric) monoidal closed categories: An arrow $f : A \to B$ is *nuclear* iff the associated $\mathbb{1} \to [A, B]$ factorises through $A^* \otimes B$, where $A^* = [A, \mathbb{1}]$, and an object A is *nuclear* if id_A is so [5]. Equivalently, $B \otimes A^* \cong [A, B]$ holds for all objects B. The nuclear objects in the category of sup-lattices and join-preserving maps are precisely the *completely distributive lattices* [5, 11, 12]. The analogue concept in the realm of $\mathcal{V}$-enriched categories, the cocomplete and completely distributive $\mathcal{V}$-categories ($\mathcal{V}$-ccd) [14], are (co)complete $\mathcal{V}$-categories for which taking suprema distributes over limits, equivalently, they are the projective objects in $\mathcal{V}\text{-}\mathsf{Sup}$. Their behaviour with respect to the monoidal structure of $\mathcal{V}\text{-}\mathsf{Sup}$ is described in the next Proposition:

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Proposition 2. *1. The tensor product in $\mathcal{V}$-Sup of two $\mathcal{V}$-ccds is $\mathcal{V}$-ccd.*

2. The nuclear objects in $\mathcal{V}$-Sup are precisely the $\mathcal{V}$-ccds.

To each arbitrary category A canonically corresponds a cocomplete one, namely $\mathbb{D}A$. Being free in $\mathcal{V}$-Sup, the latter is also $\mathcal{V}$-ccd. To obtain more examples of (completely distributive) cocomplete $\mathcal{V}$-categories associated to A, consider the Isbell adjunction $[A^{\mathsf{op}}, \mathcal{V}] \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} [A, \mathcal{V}]^{\mathsf{op}}$. Taking the fixed points of this adjunction produces a $\mathcal{V}$-category $\mathbb{I}(A)$ into which A embeds, known as the *Isbell completion*, the categorical analogue of the MacNeille completion by cuts of a poset. As such, $\mathbb{I}(A)$ is a (complete and) cocomplete $\mathcal{V}$-category, hence an object of $\mathcal{V}$-Sup. When $\mathcal{V}$ is the two-element quantale, A is just an ordered set, and $\mathbb{I}(A)$ is a complete sup-lattice, which moreover is completely distributive if and only if the negation of the underlying order of A is a regular relation [2]. Regularity is a concept definable not only for relations; it can apply to arrows in an arbitrary category [9]. We seek to generalise this result to $\mathcal{V}$-categories. However, a quantale has only tensor product and internal hom. Properly handling negation in a quantale requires extra assumptions. We shall assume that $\mathcal{V}$ is a Girard quantale, the posetal analogue of a $*$-autonomous category. Then taking internal homs into the cyclic dualising element of $\mathcal{V}$ determines a negation operation $\neg$ on $\mathcal{V}$, in particular, on all $\mathcal{V}$-valued relations. Under these assumptions, we obtain the sought generalisation:

Theorem 3. *Let $\mathcal{V}$ be a Girard integral quantale and $\mathbb{I}(A)$ the Isbell completion of a $\mathcal{V}$-enriched category A. Then the following are equivalent:*

1. *$\mathbb{I}(A)$ is completely distributive (as a cocomplete $\mathcal{V}$-category), that, is, $\mathbb{I}(A)$ is a nuclear object in the $*$-autonomous category $\mathcal{V}$-Sup.*
2. *Negation $\neg A(-,-)$ of the $\mathcal{V}$-hom of $\mathcal{A}$ is regular as a $\mathcal{V}$-relation.*

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