On the structure of balanced residuated posets

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Abstract

A residuated poset is a structure of the form $(A, \leq, \cdot, 1, \backslash, /)$ such that $(A, \leq)$ is a poset, $(A, \cdot, 1)$ is a monoid and $x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z$. It is called *balanced* if it satisfies the identity $x/x = x\backslash x$, or equivalently all positive idempotents are central (i.e., $1 \leq x = x^2 \implies x \cdot y = y \cdot x$). In this case we denote the term x/x by 1_x and call it a local identity since it satisfies $1_x \cdot x = x = x \cdot 1_x$.

We show that any balanced residuated poset can be decomposed into components $C_x = \{y : 1_y = 1_x\}$ and two families of maps from which the original residuated poset can be reconstructed. If the balanced residuated poset satisfies the identities $1_{x \cdot y} = 1_x \cdot 1_y = 1_{x/y} = 1_{x\backslash y}$ then it decomposes as a Płonka-style sum over a semilattice direct system of integral residuated posets. This structure theory generalizes the results in [1] where the residuated posets were assumed to be involutive and locally integral, hence square-decreasing.

The construction of Płonka sums from finite families of finite integral involutive residuated posets has been implemented in Python. To allow for a convenient specification of semilattice direct systems of maps, we define dual partial function systems over sets of indecomposable residuated posets. If the partial functions are assumed to be continuous with respect to Stone spaces on their domain and codomain then the components of the semilattice direct systems are Boolean products over these Stone spaces.

The glueing construction in [2] for finite commutative idempotent involutive residuated lattices produces lattice-ordered algebras rather than po-algebras. In the setting of involutive residuated lattices without finiteness, commutativity or idempotence, we show that the glueing of two integral components, over an isomorphic filter and ideal in the respective component, again produces an involutive residuated lattice. Ongoing research aims to extend this result to Płonka-style sums of balanced residuated lattices.

The results reported here are joint research with Stefano Bonzio, José Gil-Férez, Adam Přenosil and Melissa Sugimoto.

References

[1] J. Gil-Férez, P. Jipsen, and S. Lodhia. The structure of locally integral involutive po-monoids and semirings. In *Relational and algebraic methods in computer science*, volume 13896 of *Lecture Notes in Comput. Sci.*, pages 69–86. Springer, Cham, 2023.

[2] P. Jipsen, O. Tuyt, and D. Valota. The structure of finite commutative idempotent involutive residuated lattices. *Algebra Universalis*, 82(4):57, 1–23, 2021.