The Tree Structure of Conservative Commutative Residuated Lattices

Nikolaos Galatos and José Gil-Férez

[1] Denver University
[2] Chapman University

Abstract

A residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is *commutative* and *idempotent* if its monoidal part $\langle A, \cdot, e\rangle$ is commutative and idempotent, that is, if it satisfies the equations $x \cdot y = y \cdot x$ and $x \cdot x \approx x$, respectively. And we call $\mathbf{A}$ a *residuated chain* if the order $\leqslant$ associated to its lattice part $\langle A, \wedge, \vee\rangle$ is total. In [1] we studied several classes of idempotent residuated chains and their generated varieties. In particular, we established a more symmetric version of Raftery's characterization theorem [2] for commutative idempotent residuated chains, obtaining also as a corollary (as in [2]) that they generate a locally finite variety.

In that work, it was instrumental the fact that the monoidal structure of any idempotent residuated lattice $\mathbf{A}$ is a unital band and the relation on A defined by $a \sqsubseteq b \iff a \cdot b = a$ is a preorder that we call the *monoidal preorder* of $\mathbf{A}$; if the product of $\mathbf{A}$ is also commutative, then $\langle A, \sqsubseteq, e\rangle$ is a unital meet-semilattice with order $\sqsubseteq$ and greatest element e. An idempotent residuated lattice is *conservative* if its monoidal preorder is total, that is, for all a, b in A, $a \cdot b \in \{a, b\}$. For instance, every idempotent residuated chain is conservative.

In the present work, we complete the study of the class of conservative commutative residuated lattices initiated in [1] — in which we gave an account of its finite members only —, and present a general structure theory for all the members of this class. We show that the lattice of every conservative commutative residuated lattice can be described as a tree in which, moreover, all its leaves are also linearly ordered.

More in detail, a *conservative tree* is a first-order structure $\mathbf{M} = (M, \sqcap, \trianglelefteq, e)$, where

(1) $(M, \sqcap)$ is a meet-semilattice that is also a tree (we will denote its order by $\sqsubseteq$ and the set of its maximal elements by M^+);
(2) every element of M is below a $\sqsubseteq$-maximal element;
(3) $(M^+, \trianglelefteq)$ is a chain with bottom element e;
(4) for every $m \in M$, the set $\nabla m := \{p \in M^+ : m \sqsubseteq p\}$ is a closed interval of $(M^+, \trianglelefteq)$.

We prove that every conservative commutative residuated lattice $\mathbf{A}$ gives rise to a conservative tree $\mathbf{M}_{\mathbf{A}}$; and that from every conservative tree $\mathbf{M}$ we can construct a conservative commutative residuated lattice $\mathbf{A}_{\mathbf{M}}$. Moreover, these correspondences are inverse to each other.

We use this representation to settle various open problems. In particular, we show that local finiteness fails for the class of conservative commutative residuated lattices, as it contains a 1-generated infinite member. We prove also that the class of conservative commutative residuated lattices has the strong amalgamation property constructing a strong amalgam for every V-formation.

References

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