Sheaf Semantics for Inquisitive Logic

Colin Zwanziger[1] and Vít Punčochář[1]

Czech Academy of Sciences, Prague, Czechia
zwanzigerc@gmail.com, puncochar@flu.cas.cz

1 Introduction

Inquisitive logic [8, 3, 1] is a logic of so-called inquisitive propositions, intended to model questions in much the same way that the propositions of non-inquisitive logic model declarations. This logic has many interesting linguistic applications [2]. First-order inquisitive logic was studied in, *e.g.*, [7], and intuitionistic inquisitive logic was introduced in [10, 11].

In our talk, we provide a categorical analysis of the main mathematical features of inquisitive logic. In particular, we give a sheaf-theoretic semantics for (higher-order, intuitionistic) inquisitive logic. This subsumes as special cases the classical possible-worlds model of inquisitive logic [12], a refinement of this based on a topological space of worlds, as well as other models with a topological flavor.

It was observed in the propositional case by [9] that the language of (intuitionistic) inquisitive logic can be identified with (intuitionistic) logic, together with a geometric modality ∇ in the sense of [6], also known as a Lawvere-Tierney modality or lax modality. Inquisitive logic is then characterized by the addition of the so-called 'split' axiom.

$$\frac{\nabla\alpha \to \phi \vee \psi}{(\nabla\alpha \to \phi) \vee (\nabla\alpha \to \psi)}\ \text{Split}$$

From the inquisitive perspective, ∇ is understood as the presupposition modality, with $\nabla\alpha$ representing the declarative proposition presupposed by the inquisitive proposition α.

2 Higher-Order Semantics

To extend Holliday's insight from the propositional setting to higher-order, we must pass from Heyting algebras and nuclei to toposes and Cartesian reflectors.

Essentially since Lawvere and Tierney, it has been known that a topos $\mathcal{E}$ equipped with with a Cartesian reflector $J : \mathcal{E} \to \mathcal{E}$ interprets intuitionistic higher-order logic with a geometric modality. The Lawvere-Tierney operator $j : \Omega \to \Omega$ in $\mathcal{E}$ induced by J interprets the geometric modality ∇. The rest of the logic is interpreted standardly in $\mathcal{E}$. Our move will be to narrow down this abstract semantics in order to validate the additional axioms of inquisitive logic.

Theorem 1. *Let $(\mathbf{C}, J)$ be a site where $\mathbf{C}$ is small and cocomplete and J is canonical. Then, $\mathbf{Set}^{\mathbf{C}^{op}}$, together with the sheafification $a : \mathbf{Set}^{\mathbf{C}^{op}} \to \mathbf{Set}^{\mathbf{C}^{op}}$ induced by J is a model of of intuitionistic higher-order inquisitive logic.*

3 Examples

Example 2. *Let W be a set (of possible worlds). Then, the singleton injection*

$$\{\cdot\} : W \rightarrowtail \mathbf{2}^W$$

induces the adjunction

$$\{\cdot\}^* \dashv \{\cdot\}_* : \mathbf{Set}^W = \mathbf{Set}^{W^{\mathrm{op}}} \rightarrowtail \mathbf{Set}^{(\mathbf{2}^W)^{\mathrm{op}}} \quad .$$

The composite $\{\cdot\}_*\{\cdot\}^*$ *is a Cartesian reflector, and thus induces a coverage of* $\mathbf{2}^W$*, which is canonical. Moreover,* $\mathbf{2}^W$ *is small and cocomplete (*i.e. *admits small joins).*

This recovers the classical model of predicate inquisitive logic. In particular, we have $\mathrm{Sub}_{\mathbf{Set}^{(\mathbf{2}^W)^{\mathrm{op}}}}(1) \cong \mathbf{2}^{(\mathbf{2}^W)^{\mathrm{op}}}$ *and* $\mathrm{Sub}_{\mathbf{Set}^W}(1) \cong \mathbf{2}^W$, i.e. *the subsingletons of* $\mathbf{Set}^{(\mathbf{2}^W)^{\mathrm{op}}}$ *and* $\mathbf{Set}^W$ *correspond respectively to downwards-closed sets of subsets of* W *and subsets of* W*, which in inquisitive logic following [12] are respectively identified with inquisitive propositions and declarative propositions.*

Example 3. *Any topological space* W *(of possible worlds), regarded as a site, satisfies the conditions of Theorem 1. Thus,* $\mathbf{Set}^{\mathcal{O}(W)^{\mathrm{op}}}$*, together with the sheafification*

$$\mathbf{Set}^{\mathcal{O}(W)^{\mathrm{op}}} \xrightarrow{a} \mathrm{Sh}(W) \hookrightarrow \mathbf{Set}^{\mathcal{O}(W)^{\mathrm{op}}}$$

is a model.

In particular, we have $\mathrm{Sub}_{\mathbf{Set}^{\mathcal{O}(W)^{\mathrm{op}}}}(1) \cong \mathbf{2}^{\mathcal{O}(W)^{\mathrm{op}}}$ *and* $\mathrm{Sub}_{\mathrm{Sh}(W)}(1) \cong \mathcal{O}(W)$*, which we might identify with answerable inquisitive propositions and verifiable declarative propositions, respectively.*

The classical model of Example 2 is recovered in the case where W *is discrete and thus* $\mathcal{O}(W) = \mathbf{2}^W$*.*

Additional examples include sheaves on a locale, and, when size issues are dealt with, sheaves on an ionad [5, 4] and sheaves on a Grothendieck topos.

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