On the implicative subreducts of subresiduated lattices

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Abstract

It is known that sub-Hilbert algebras are the implicative subreducts of subresiduated lattices. In this work we give a new proof of this property by using ideas employed to represent weak Heyting algebras. We also study the lattice of relative congruences of sub-Hilbert algebras and we give a quasi-equational description of the quasivariety of sub-Hilbert algebras generated by the class of its totally ordered members.

Subresiduated lattices were introduced by Epstein and Horn [8] with the aim to study certain propositional logics defined in a language without classical implication but with a connective of implication which is called strict implication. The logics studied in [8] are examples of subintuitionistic logics, i.e., logics in the language of intuitionistic logic that are defined semantically by using Kripke models, in the same way as intuitionistic logic is defined, but without requiring of the models some of the properties required in the intuitionistic case [5, 6].

A subresiduated lattice (sr-lattice for short) [6, 8] is a pair (A, D), where A is a bounded distributive lattice, D is a bounded sublattice of A and for every $a, b \in A$ there exists the maximum of the set $\{d \in D : a \wedge d \leq b\}$, which is denoted by $a \rightarrow b$. This pair can be regarded as an algebra $(A, \wedge, \vee, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ where $D = \{a \in A : 1 \rightarrow a = a\}$. The class of sr-lattices properly contains the variety of Heyting algebras. It follows from [8, Theorem 1] that the class of sr-lattices forms a variety. A different equational base for this variety was given in [6], where this variety is presented as a subvariety of the variety of weak Heyting algebras.

Recall that S4-algebras are Boolean algebras with a unary operator $\Box$ in the language that satisfies the identities $\Box 1 = 1$, $\Box(x \wedge y) = \Box x \wedge \Box y$, $\Box x \leq x$ and $\Box x \leq \Box(\Box x)$. We say that an algebra $(A, \wedge, \vee, \rightarrow, \neg, 0, 1)$ is a Boolean subresiduated lattice (Boolean sr-lattice for short) if $(A, \wedge, \vee, \neg, 0, 1)$ is a Boolean algebra and $(A, \wedge, \vee, \rightarrow, 0, 1)$ is a sr-lattice. If $(A, \wedge, \vee, \rightarrow, \neg, 0, 1)$ is a Boolean sr-lattice then $(A, \wedge, \vee, \neg, \Box, 0, 1)$ is a $S4$-algebra, where $\Box x := 1 \rightarrow x$. Conversely, if $(A, \wedge, \vee, \neg, \Box, 0, 1)$ is a S4-algebra then $(A, \wedge, \vee, \rightarrow, \neg, 0, 1)$ is a Boolean sr-lattice, where $x \rightarrow y := \Box(\neg x \vee y)$. Moreover, the variety of boolean sr-lattices is term equivalent to the variety of S4-algebras. It can be also showed that the variety of sr-lattices coincides with the class of $\{\wedge, \vee, \rightarrow, 0, 1\}$-subreducts of Boolean sr-lattices.

A sub-Hilbert algebra [3] is an algebra $(A, \rightarrow, 1)$ of type (2,0) which satisfies the following quasi-equations:

- $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$,
- $x \rightarrow x = 1$,
- $x \rightarrow 1 = 1$,
- if $x \rightarrow y = 1$ and $y \rightarrow x = 1$, then $x = y$,

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- $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$.

We write sHA to indicate the class of sub-Hilbert algebras, which properly contains the variety of Hilbert algebras. The class sHA is a quasivariety which is not a variety [4]. Moreover, in sub-Hilbert algebras the binary relation $\leq$, defined by $a \leq b$ if and only $a \to b = 1$, is a partial order and 1 is the last element with respect to this order [4]. In [4] it was also proved that sub-Hilbert algebras are the implicative subreducts of sr-lattices, property which generalizes the fact that Hilbert algebras are the implicative subreducts of Heyting algebras [1, 7]. This result is proved by using properties of implicative filters of sub-Hilbert algebras, where an implicative filter of a sub-Hilbert A is a subset F of A such that $1 \in F$ and $b \in F$ whenever $a, a \to b \in F$.

In the present work we prove, following an alternative path to that given in [4] and motivated by certain constructions developed in [3, 6] for some classes of algebras, that sub-Hilbert algebras are the implicative subreducts of sr-lattices. More precisely, given $A \in$ sHA we show the following two facts: 1) it is possible to define a binary relation R on the set $\mathrm{IF}(A)$ of implicative filters of A which induces a binary operation $\Rightarrow_R$ on the set $\mathrm{IF}(A)^+$ of upsets of $(\mathrm{IF}(A), \subseteq)$ such that $(\mathrm{IF}(A)^+, \cap, \cup, \Rightarrow_R, \emptyset, \mathrm{IF}(A))$ is a sr-lattice; 2) there exists a monomorphism from A to $(\mathrm{IF}(A)^+, \Rightarrow_R, \mathrm{IF}(A))$. We also show that for every $A \in$ sHA, the lattice of relative congruences of A is order isomorphic to the lattice of open implicative filters of A, where an implicative filter F of A is said to be open if $1 \to a \in F$ whenever $a \in F$. Moreover, we study properties of the irreducible open implicative filters [1]. Finally, motivated by some results given in [2, 9], we apply properties of irreducible open implicative filters in order to give a quasi-equational description of the quasivariety of sHA generated by the class of its totally ordered members.

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[1] Let $A \in$ sHA. A proper open implicative filter P of A is called irreducible when for all F_1 and F_2 open implicative filters, if $P = F_1 \cap F_2$, then $P = F_1$ or $P = F_2$.