Canonical Extensions of Quantale Enriched Categories

Alexander Kurz[1] and Apostolos Tzimoulis[2]

[1] Chapman University, CA, USA
[2] Vrije Universiteit, Amsterdam, The Netherlands

Introduction. We extend canonical extensions from the ordered to the quantale enriched setting. We pay particular attention to *non-commutative* quantales and develop our work in a language that stays as faithful as possible to both order theory [4] and category theory [10]. The reason is not only to make our work accessible to both communities: In our own ongoing work, we need to have easy access to general category theoretic results and to an algebraic language in the style of lattice theory and logic.

Related Work. [2] defines the MacNeille completion of a relation $\mathrm{I} : X \times A \to 2$. Our interest stems from generalising Formal Context Analysis [6] with its applications to data bases and data analysis to the fuzzy and many-valued setting [9, 3]. The MacNeille completion of quantale enriched categories has been studied in [11, 7, 5].

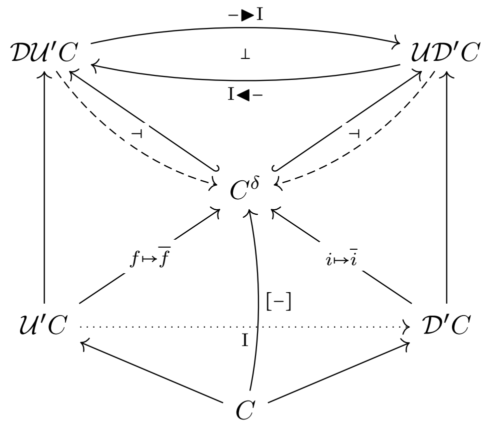
Quantales. A quantale $(\Omega, \sqsubseteq, \bigsqcup, e, \cdot,)$ is a complete join semilattice $(\Omega, \sqsubseteq, \bigsqcup)$ and a monoid $(\Omega, e, \cdot)$ in which multiplication distributes over joins. We write top as $\top$ and bottom as $\bot$. Since Ω is complete it also has meets $\bigsqcap$. Multiplication has a left-residual $\triangleleft$ and the right-residual $\triangleright$ defined as $b \sqsubseteq a \triangleright c \Leftrightarrow a \cdot b \sqsubseteq c \Leftrightarrow a \sqsubseteq c \triangleleft b$.

Examples. (a) The *two-chain* $2 = \{0 \sqsubseteq 1\}$ is a commutative quantale.
(b) The *Lawvere quantale* $[0, \infty]$ is a subset of the extended real numbers. It is ordered by $\geq$ with top $\top = 0$ and has $+$ as multiplication. The residual is truncated minus $a \triangleleft b = a \dot{-} b$.
(c) The *quantale of languages* $\mathcal{P}(\Sigma^*)$ is given wrt a set Σ and has as elements subsets of the set of finite words Σ^*. Multiplication is $L \cdot L' = \{vw \mid v \in L, w \in L'\}$ where vw denotes the concatenation of the words. The residuals are given by $L \triangleright M = \{w \in \Sigma^* \mid \forall v \in L \,.\, vw \in M\}$ and $M \triangleleft L = \{w \in \Sigma^* \mid \forall v \in L \,.\, wv \in M\}$.

Quantale Spaces. We call a category enriched over a quantale a *quantale space*. Enrichment over 2 gives preorders, enrichment over $[0, \infty]$ gives generalized metric spaces [8], enrichement over $\mathcal{P}(\Sigma^*)$ gives generalized non-deterministic automata (without designated initial and final states) [1].

Weighted Downsets and Upsets. A *relation* (also known as bimodule, profunctor, distributor) $R : X \looparrowright Y$ between quantale spaces X and Y is a function $X \times Y \to \Omega$ satisfying $X(x', x) \cdot R(x, y) \sqsubseteq R(x', y)$ and $R(x, y) \cdot Y(y, y') \sqsubseteq R(x, y')$. A *presheaf* (or weighted downset) $\phi \in \mathcal{D}X$ is a relation $X \looparrowright 1$. A *co-presheaf* (or weighted upset) $\psi \in \mathcal{U}Y$ is a relation $1 \looparrowright Y$. The homs are defined by $\mathcal{D}X(\phi, \phi') = \bigsqcap_{x \in X} (\phi x \triangleright \phi' x)$ and $\mathcal{U}A(\psi, \psi') = \bigsqcap_{a \in A} (\psi a \triangleleft \psi' a)$.

Canonical Extension. The canonical extension C^δ of a quantale space C is the MacNeille completion of the relation $\mathrm{I} : \mathcal{U}'C \looparrowright \mathcal{D}'C$ given by $\mathrm{I}(f, i) = \bigsqcup_c f(c) \cdot i(c)$, that is, the set of fixed points of the adjunction given by $\phi \blacktriangleright \mathrm{I} = \bigsqcap_f \phi(f) \triangleright \mathrm{I}(f, -)$ and $\mathrm{I} \blacktriangleleft \psi = \bigsqcap_i \mathrm{I}(-, i) \triangleleft \psi(i)$.



Here $\mathcal{U}'C$ and $\mathcal{D}'C$ are subsets of $\mathcal{U}C$ and $\mathcal{D}C$ containing the representable (co)presheaves. The paradigmatic example is the set of all "weighted" filters f and "weighted" ideals i.

Theorem Let $f \in \mathcal{U}'C$ and $i \in \mathcal{D}'C$. Then C^δ is compact in the sense that[1] $C^\delta(\lim_f[-], \mathrm{colim}_i[-]) = \mathrm{I}(f, i)$. Moreover, every $(\phi, \psi) \in C^\delta$ is the colimit of a limit of C and the limit of a colimit of C.

In our talk, we will introduce an algebraic calculus for reasoning in quantale enriched categories, present the canonical extension construction, provide some examples, and discuss the extensions of functors to the canonical extensions.

References

[1] Renato Betti and Stefano Kasangian. A quasi-universal realization of automata, 1982.

[2] G. Birkhoff. *Lattice Theory*. American Mathematical Society, 1948.

[3] Marcel Boersma, Krishna Manoorkar, Alessandra Palmigiano, Mattia Panettiere, Apostolos Tzimoulis, and Nachoem Wijnberg. Outlier detection using flexible categorization and interrogative agendas. *Decision Support Systems*, 180, 2024.

[4] J. Michael Dunn, Mai Gehrke, and Alessandra Palmigiano. Canonical extensions and relational completeness of some substructural logics. *Journal of Symbolic Logic*, 70(3), 2005.

[5] Soichiro Fujii. Completeness and injectivity. *Topology and its Applications*, 301, 2021.

[6] Bernhard Ganter and Rudolf Wille. *Formal Concept Analysis - Mathematical Foundations*. Springer, 1999.

[7] Richard Garner. Topological functors as total categories. *Theory and Applications of Categories*, 29(15), 2014.

[8] F.W. Lawvere. Metric spaces, generalized logic and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano*, XLIII, 1973. Republished in *Reprints in Theory Appl. Categ.*

[9] Dusko Pavlovic. Quantitative concept analysis. In *ICFCA 2012*.

[10] Isar Stubbe. Categorical structures enriched in a quantaloid: categories, distributors and functors. *Theory and Applications of Categories*, 14(1), 2005.

[11] Simon Willerton. Tight spans, Isbell completions and semi-tropical modules. *Theory and Applications of Categories*, 28(22):696–732, 2013.

[1] $\mathrm{lim_f}$ refers to the limit weighted by f and $\mathrm{colim_i}$ to the colimit weighted by i.