Semi-prelinear Residuated Lattices

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Abstract

The origin of residuated lattices is in Mathematical Logic without contraction. They have been investigated by Krull [11], Dilworth [4], Balbes and Dwinger [1] and Pavelka [12]. In [9], Idziak proved that the class of residuated lattices is equational. Several authors have introduced different sub-classes of residuated lattices such as De Morgan residuated lattices [8], quasicomplemented residuated lattices [13], divisible residuated lattices [2], semi-divisible residuated lattices [2], MTL-algebra [5], BL-algebra and G-algebra [6] to name only these.

During the last decades, the fuzzy logic has become very popular, mainly because of its applicational aspects. In the framework of "soft computing", continuous triangular t-norms are used as "conjunction" and the corresponding residuum as "implication" to combine fuzzy sets with membership values in $[0, 1]$. This has a great importance in applicational aspects, particularly in fuzzy control, uncertain modeling, graph theory, data visualization and analysis. BL-algebras have been invented by Hajek [6] in order to provide an algebraic proof of the completeness of basic fuzzy logic (BL for short), the logic of continuous t-norms and their residua. However, a sufficient and necessary condition for a t-norm to have a residuated implication is left-continuity; hence it makes sense to consider fuzzy logics based not on continuous t-norms but on left-continuous t-norms. To this end, Esteva and Godo proposed in [5] a new logic, called MTL, as the basic fuzzy logic in this more general sense. The proposal was successful when Jenei and Montagna proved in [10] that MTL is indeed the logic of all left-continuous t-norms and their residua. The algebraic models of MTLs are MTL-algebras, the divisibility condition $x \odot (x \to y) = x \wedge y$ does not hold, so they are residuated lattices verifying only the prelinearity condition $(x \to y) \vee (y \to x) = 1$. MTL-algebras have been widely studied in the literature [5, 3]. It has been proven that anintegral idempotent residuated lattice is a Heyting algebra, hence idempotency is a very strong notion in the residuated lattice setting. Various special residuated lattices are now used as the main structure of truth values in fuzzy set theory and are subject to algebraic investigation.

In the present paper, we introduce the concept of semi-idempotent residuated lattices, provide new characterizations and establish many of their important properties. In addition, introduce a new subclass of residuated lattices called semi-prelinear residuated lattices. This is done by replacing the prelinearity axiom $(x \to y) \vee (y \to x) = 1$ by a weaker axiom: $(x' \to y') \vee (y' \to x') = 1$ where $x' = x \to 0$, called semi-prelinearity equation. This study was motivated in part by the fact that a similar approach was used to treat the concept of semi-divisibility by weakening the divisibility axiom $x \odot (x \to y) = x \wedge y$ to: $[x' \odot (x' \to y')]' = (x' \wedge y')'$. We study semi-prelinear residuated lattices and establish the links with several subclasses of residuated lattices. Many results similar to those obtained for MTL-algebras are obtained. The different situations depicted with prime ideals and filters show the gap between semi-prelinearity and prelinearity. In order to stress the divide

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between the two classes of residuated lattices, we point at some important results that hold in prelinear settings but not in semi-prelinear ones (see., e.g., [16, Theorem 4]). Finally, we extend the work of Belohlavek and Vychodil [14] by computing the numbers of residuated lattices such as semi-prelinear, semi-divisible, De Morgan, Stonean and semi-idempotent up to order 12.

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