Colimits of Heyting Algebras through Esakia Duality

Rodrigo Nicolau Almeida

Institute of Logic, Language and Computation – University of Amsterdam

In contrast to Boolean algebras and Distributive lattices, the variety of Heyting algebras is not locally finite, and in fact, none of its free finitely generated algebras are finite. The associated difficulty of understanding the free algebras has motivated a wealth of research into describing such algebras. Such investigations were carried out by Bellissima [3], Grigolia [6], [7] and Urquhart [8], as well as later by Ghilardi [5]. These have allowed semantic proofs of several key facts regarding this category of algebras: the fact that finitely presented Heyting algebras are bi-Heyting algebras being a prime example. Despite this wealth of work, a description of the free Heyting algebra on any number of generators seems to not have been presented in the literature. In this talk (based on an available preprint [1]) we generalize Ghilardi's [5] construction of the free Heyting algebra generated by a finite distributive lattice to any distributive lattice. The key technical tool employed is Priestley duality, as well as the following adaptations of Ghilardi's construction:

Definition 1. Let X, Y, Z be Priestley spaces, and $g : X \to Y$ and $f : Y \to Z$ be Priestley morphisms. We say that f is *open relative to g* (g-open for short) if it satisfies the following:

$$\forall a \in X, \forall b \in Y, (f(a) \leq b \implies \exists a' \in X, (a \leq a' \ \&\ g(f(a')) = g(b)). \tag{*}$$

Given $S \subseteq X$ a closed subset, we say that S is *g-open* (understood as a poset with the restricted partial order relation) if the inclusion is itself g-open.

Definition 2. Let $g : X \to Y$ be a map between Priestley spaces. Then consider

$$V_g(X) := \{C \subseteq X : C \text{ is closed, rooted and } g\text{-open } \},$$

with the topology given by a subbasis consisting of sets of the form

$$[U] = \{C \in V_g(X) : C \subseteq U\} \text{ and } \langle V \rangle = \{C \in V_g(X) : C \cap V \neq \emptyset\},$$

where U, V are clopen subsets of X.

The following can then be shown:

Proposition 3. *Given $g : X \to Y$ a Priestley morphism, the space $(V_g(X), \preceq)$ is a Priestley space, equipped with a continuous surjection $r_g : V_g(X) \to X$ sending each rooted subset to its root.*

The construction V_g enjoys a specific universal property:

Lemma 4. Given a Priestley map $g : X \to Y$, the construction V_g enjoys the following property: given a Priestley space Z with a g-open continuous and order-preserving map $h : Z \to X$, there exists a unique r_g-open, continuous and order-preserving map h' such that the triangle in Figure 1 commutes.

$$\begin{tikzcd} Z \arrow[rr, "h'"] \arrow[dr, "h"'] & & V_g(X) \arrow[dl, "r"] \\ & X & \end{tikzcd}$$

Figure 1: Commuting Triangle of Priestley spaces

Definition 5. Let $g : X \to Y$ be a Priestley morphism. The *g-Vietoris complex* over X $(V^g_\bullet(X), \leq_\bullet)$, is a sequence

$$(V_0(X), V_1(X), ..., V_n(X))$$

connected by morphisms $r_i : V_{i+1}(X) \to V_i(X)$ such that:

1. $V_0(X) = X$;
2. $r_0 = g$
3. For $i \geq 0$, $V_{i+1}(X) := V_{r_i}(V_i(X))$;
4. $r_{i+1} = r_{r_i} : V_{i+1}(X) \to V_i(X)$ is the root map.

We denote the projective limit of this family by $V_G^g(X)$, and omit it when g is the terminal map to $\mathbf{1}$.

Theorem 6. *The assignment V_G is a functor mapping the category* **Pries** *of Priestley spaces and Priestley morphisms to the category* **Esa** *of Esakia spaces; indeed it is the right adjoint of the inclusion.*

As applications, we obtain new proofs of old results, as well as some new facts: (1) a description of free Heyting algebras on any number of generators is given; (2) a description of coproducts of Heyting algebras is given, and it is shown that the category of Heyting algebras is co-distributive; (3) A description of pushouts of Heyting algebras is given, and it is shown directly that the coprojections of Heyting algebras to the pushout are injective (yielding, as a corollary, the amalgamation property).

We also consider two generalizations of these results:

1. We consider the construction obtained when restricting to specific subvarieties of **HA**, such as KC-algebras and LC-algebras (often called "Gödel algebras"), and show that adaptations of the above ideas yield descriptions of the free algebras in these varieties.
2. We study the category of image-finite posets and p-morphisms and its relationship to the category of posets. We show that a similar adjunction holds here. This connects with recent work by de Berardinis and Ghilardi [4], and provides a generalization of the n-universal model for arbitrary finite posets.

We also highlight some connections to coalgebraic representations of intuitionistic modal logic, which we investigate in depth in a paper with Nick Bezhanishvili [2]. We conclude by pointing some further avenues of exploration, as well as some questions left open by the above research:

Problem 7. Is $V_g(X)$ an Esakia (bi-Esakia) space whenever X is an Esakia (bi-Esakia) space?

Problem 8. Does the inclusion of $\mathbf{Pos}_p$, the category of posets with p-morphisms, into **Pos** admit a right adjoint?

References

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