Characterizing finite measurable Boolean algebras*

Philippe Balbiani[a] , Quentin Gougeon[a†], and Tinko Tinchev[b]

[a]Institut de recherche en informatique de Toulouse
CNRS-INPT-UT3
Toulouse, France
philippe.balbiani@irit.fr, quentin.gougeon@irit.fr

[b]Faculty of Mathematics and Informatics
Sofia University St. Kliment Ohridski
Sofia, Bulgaria
tinko@fmi.uni-sofia.bg

In our daily life, we are used to compare things. We sort physical objects from smaller to bigger, or propositions from less likely to more likely. These relations contribute to our intuitive understanding of reality, and are naturally represented by a pre-order on a Boolean algebra. Such framework is typically *qualitative*, as two elements can only be related in three possible manners: smaller, bigger, or equivalent – without any consideration of degree or magnitude. By contrast, human beings also managed to quantify some of their intuitions through *measurement*, with examples including length, volume, temperature, and probabilities. Quantitative reasoning is a core component of scientific inquiry, and its mathematical foundations have been studied extensively in [KLTS71]. Formally, a *measure* on a **finite**[1] Boolean algebra $B = (2^X, \cap, \cup, \cdot^c, 0, 1)$ is a map $\mu : B \to [0, \infty]$ satisfying $\mu(0) = 0$ and $\mu(a \cup b) = \mu(a) + \mu(b)$ whenever $a \cap b = 0$. We call μ *bounded* if in addition we have $\mu(a) < \infty$ for all $a \in B$. Obviously, a measure μ always induces a binary relation $\preceq_\mu$ on B, defined by $a \preceq_\mu b \iff \mu(a) \leq \mu(b)$. Relations of the form $\preceq_\mu$ will be called *measurable*, and *bounded measurable* in case μ is a bounded measure. So there is a direct bridge from quantitative to qualitative comparison, but the other way around is more limited, and this raises the question of which conditions on a binary relation $\preceq$ are necessary and sufficient for $\preceq$ to be (bounded) measurable. In the case of bounded measures, this problem was solved by Kraft, Pratt and Seidenberg in their 1959 paper [KPS59], and later rewritten by Scott [Sco64] in a clearer manner. We present their conditions below. Given $x \in X$ and $a_1, \dots, a_m \in B$, we write $\texttt{count}_x(a_1, \dots, a_m) := \{i \in [1, m] : x \in a_i\}$.

Theorem 1. *A binary relation $\preceq$ on B is bounded measurable if and only if the following conditions are satisfied, for all $m \geq 1$ and for all $a, b, a_1, \dots, a_m, b_1, \dots, b_m \in B$:*

- **Positivity:** $0 \preceq a$;
- **Comparability:** $a \preceq b$ *or* $b \preceq a$;
- **Cancellation:** *if* $\texttt{count}_x(a_1, \dots, a_m) = \texttt{count}_x(b_1, \dots, b_m)$ *for all* $x \in X$ *and* $a_i \preceq b_i$ *for all* $i \in [1, m-1]$, *then* $b_m \preceq a_m$.

However, this result is not fully satisfying for a number a reasons, related to the cancellation conditions. First, they involve the high-level operator $\texttt{count}_x$, and even though they can be rewritten in a purely Boolean manner [Seg71], they remain quite awkward to read and compute. Second, they come in infinite number, and thus fail to provide a finite axiomatization for various logics of measure, see for instance [Seg71, Gär75, vdH96]. It is surprising, perhaps, that this result has never been improved in sixty years, nor proved to be optimal. In this work, we break this uncomfortable status quo by proposing the following new characterization.

Theorem 2. *A binary relation $\preceq$ on B is bounded measurable if and only if the following conditions are satisfied, for all $a, b, c, d \in B$:*

*The preprint on which this talk is based can be found here: https://hal.science/hal-04544145.
†Speaker.
[1]The case of infinite Boolean algebras is much more complicated, so we only consider the finite case.

- **Comparability:** *$a \preceq b$ or $b \preceq a$;*
- **Linearity:** *if $a \cap c = 0$ and $a \cup c \preceq b \cup d$ and $d \preceq c$, then $a \preceq b$.*

Let us briefly sketch the proof of Theorem 2. The strategy for the right-to-left implication is to derive the conditions of Theorem 1 from comparability and linearity. Positivity follows from linearity with $a = 0$ and $b = c = d$. For cancellation, assume that $\mathtt{count}_x(a_1, \ldots, a_m) = \mathtt{count}_x(b_1, \ldots, b_m)$ for all $x \in X$, and that $a_i \preceq b_i$ for all $i \in [1, m-1]$. Consider for a moment the case where $a_1, \ldots, a_m, b_1, \ldots, b_m$ are all pairwise disjoint. Then, the counting assumption yields $b_1 \cup \cdots \cup b_m \preceq a_1 \cup \cdots \cup a_m$, and by applying linearity $m-1$ times we arrive at $b_m \preceq a_m$. This does not work in the general case, because when $\mathtt{count}_x(a_1, \ldots, a_m) \geq 2$, the large union $a_1 \cup \cdots \cup a_m$ fails to keep track of the different repetitions of x. We can nonetheless bypass this issue, and fall back to the previous case, by 'duplicating' the elements of X. In a critical lemma, we show that we can introduce equivalent copies $x^1, \ldots, x^{2m}$ of every $x \in X$, in a way that preserves positivity, comparability, and a weaker version of linearity. We then tweak the sets $a_1, \ldots, a_m, b_1, \ldots, b_m$ by replacing their members with corresponding copies, so that one copy never occurs twice (see the example below).

$$
\begin{array}{ll|ll}
a_1 = \{x, z\} & b_1 = \{x, y\} & a_1^* = \{x^1, z^1\} & b_1^* = \{x^5, y^5\} \\
a_2 = \{x, y\} & b_2 = \{z\} & a_2^* = \{x^2, y^1\} & b_2^* = \{z^5\} \\
a_3 = \{z\} & b_3 = \{x, y, z\} & a_3^* = \{z^2\} & b_3^* = \{x^6, y^6, z^6\} \\
a_4 = \{x\} & b_4 = \{x\} & a_4^* = \{x^3\} & b_4^* = \{x^7\}
\end{array}
$$

It then suffices to apply the previous reasoning to the sets $a_1^*, \ldots, a_m^*, b_1^*, \ldots, b_m^*$.

We also address the case of arbitrary measurable relations.

Theorem 3. *A binary relation $\preceq$ on B is measurable if and only if the following conditions are satisfied, for all $a, b, c, d \in B$:*

- **Comparability:** *$a \preceq b$ or $b \preceq a$;*
- **Transitivity:** *$a \preceq b$ and $b \preceq c$ implies $a \preceq c$;*
- **Monotonicity:** *$a \subseteq b$ implies $a \preceq b$;*
- **Bounded Linearity:** *if $1 \not\preceq c$ and $a \cap c = 0$ and $a \cup c \preceq b \cup d$ and $d \preceq c$, then $a \preceq b$.*

Finally, we observe that the conditions of Theorem 2 and Theorem 3 can be checked in space logarithmic in the size of B. In the case of bounded measurable relations, this is a direct improvement on the polynomial space algorithm of Kraft, Pratt and Seidenberg [KPS59].

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