From higher-order rewriting systems to higher-order categorial algebras and higher-order Curry-Howard isomorphisms

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In proof theory and programming language theory, the Curry-Howard correspondence explains the direct relationship between computer programs and mathematical proofs. Curry, in [3], was the first to recognize the analogy between combinatory logic and the axioms of a Hilbert-style deductive system for positive propositional logic. Later, Howard [4] observed a similar formal analogy between the lambda calculus and the proof rules of a Gentzen-style natural deduction system for propositional logic. The Curry-Howard correspondence associates each proof in intuitionistic logic with a term in Curry's combinatory logic or Church's lambda calculus. This correspondence, also called *proofs-as-programs*, connects intuitionistic logic proofs with terms in combinatory logic or lambda calculus. Essentially, it reveals that deduction systems and computation models are fundamentally the same mathematical entities. The Curry-Howard correspondence sparked research leading to dual-purpose formal systems—serving as both proof systems and typed functional programming languages. Examples include Martin-Löf's intuitionistic type theory [5] and Coquand's Calculus of Constructions [2]. These systems treat proofs as regular objects, allowing properties to be declared about proofs, akin to any other program—an area known as modern type theory. In particular, the Homotopy Type Theory (HoTT) [6] is a new field of mathematical study that combines various aspects of type theory and homotopy theory, incorporating ideas from algebraic topology and homological algebra for the examination of formal derivation systems.

The work to be presented here is based on an ongoing study [1], which will serve as the foundation. In this work, we consider a signature Σ, a set of variables X, and a fixed set $\mathcal{A}$ of rewriting rules. This set $\mathcal{A}$ consists of pairs $\mathfrak{p} = (M, N)$, where M and N are terms in the free Σ-algebra $\mathrm{T}_\Sigma(X)$. These pairs are used to form a set of admissible rewriting rules in a formal derivation system $\boldsymbol{\mathcal{A}} = (\Sigma, X, \mathcal{A})$, which leads to the concept of a path between terms. A path is a finite sequence of terms in which, at each step, a rewriting rule is applied. In other words, the $(i+1)$-th term is obtained by substituting subterm M in the i-th term with term N. This object is understood as a simplified version of a proof where, at each step, a derivation rule admitted in the system is used. This leads to a category of paths, where the objects are terms in $\mathrm{T}_\Sigma(X)$, and the morphisms are paths between terms. Furthermore, it is shown that the set of paths for a rewriting system $\boldsymbol{\mathcal{A}}$ has the structure of a Σ-algebra and is equipped with an artinian order that specifies the complexity of the path and aids in the inductive study of these objects.

Next, we consider $\Sigma^{\boldsymbol{\mathcal{A}}}$, an extension of the original signature Σ, which includes both categorical operations and rewriting rules from $\mathcal{A}$. For this extension, it is proven that the set of paths has the structure of a partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra. With the assistance of the artinian order on paths, each path is associated with a term in the free $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra $\mathrm{T}_{\Sigma^{\boldsymbol{\mathcal{A}}}}(X)$, akin to the Curry-Howard construction. This term captures the syntactic derivation that occurs in each path, except for the ordering in the derivation that occurs in parallel. The study of the kernel of the Curry-Howard application reveals that it is a closed $\Sigma^{\boldsymbol{\mathcal{A}}}$-congruence, which allows for the algebraic study of the respective quotient of paths. It has been shown that the quotient of paths by the kernel of the Curry-Howard application has the structure of a category, a partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra, and a partially ordered set with an artinian partial order. This allows for the preservation of the inductive study of path classes. Moreover, there is a strong relationship between the categorical and algebraic structures, as the operations from the original signature Σ act as functors on the categorical structure. The fundamental result establishes that this quotient structure is the free partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebra for a variety $\boldsymbol{\mathcal{V}}$ of partial $\Sigma^{\boldsymbol{\mathcal{A}}}$-algebras, subject to equations related to both the categorical and algebraic structures, that is, a Curry-Howard isomorphism type result.

In the second part of this work, we generalize the previous results by considering second-order rewriting systems $\boldsymbol{\mathcal{A}}^{(2)} = (\boldsymbol{\mathcal{A}}^{(1)}, \mathcal{A}^{(2)})$, where $\boldsymbol{\mathcal{A}}^{(1)}$ is a rewriting system and $\mathcal{A}^{(2)}$ is a set of second-order rewriting rules. This helps us introducing the notion of second-order paths, in analogy to homotopies in topological spaces. Analogous results to those in the first part are established, thanks to the definition of a second-order Curry-Howard mapping. Particularly, for a second-order categorical signature $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$, a quotient set with the structure of a 2-category, a partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra, and a partially ordered set with an artinian partial preorder has been constructed. The fundamental result establishes that this quotient structure is the free partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebra for a variety $\boldsymbol{\mathcal{V}}^{(2)}$ of second-order partial $\Sigma^{\boldsymbol{\mathcal{A}}^{(2)}}$-algebras, subject to equations related to the 2-categorical and algebraic structure. In other words, we obtain a second-order Curry-Howard isomorphism for second-order rewriting systems.

This will ultimately lead, in future versions of this work, to the development of a theory aimed at investigating the relationship between ω-rewriting systems and ω-categorial algebras through higher-order Curry-Howard isomorphisms.

References

[1] J. Climent Vidal and E. Cosme Llópez. *From higher-order rewriting systems to higher-order categorial algebras and higher-order Curry-Howard isomorphisms.* ArXiv https://arxiv.org/abs/2402.12051, 2024.

[2] T. Coquand and G. Huet. *The calculus of constructions.* PhD thesis, INRIA, 1986.

[3] H. B. Curry, R. Feys, W. Craig, J. R. Hindley, and J. P. Seldin. *Combinatory logic*, volume 1. North-Holland Amsterdam, 1958.

[4] W. A. Howard. The formulae-as-types notion of construction. *To H. B. Curry: essays on combinatory logic, lambda calculus and formalism*, 44:479–490, 1980.

[5] P. Martin-Löf and G. Sambin. *Intuitionistic type theory*, volume 9. Bibliopolis Naples, 1984.

[6] The Univalent Foundations Program. *Homotopy type theory: Univalent foundations of mathematics.* ArXiv https://arxiv.org/abs/1308.0729, 2013.