Semicartesian categories of relations

Gejza Jenča[1]* and Bert Lindenhovius[2]†

[1] Slovak University of Technology, Bratislava, Slovakia
gejza.jenca@stuba.sk
[2] Mathematical Institute of the Slovak Academy of Sciences, Bratislava, Slovakia
lindenhovius@mat.savba.sk

Quantization is the process of generalizing mathematical structures to the noncommutative setting. Many quantum phenomena have classical counterparts, and can often be modelled by quantized versions of the mathematical structures modelling these classical counterparts. Recently, several mathematical structures have been quantized via a quantization method based on Weaver's notion of a *quantum relation* between von Neumann algebras [13], which he distilled from his work with Kuperberg on the quantization of metric spaces [12]. Quantum relations can be regarded as noncommutative versions of ordinary relations, and admit a rich relational calculus that allows us to generalize concepts such as symmetric, antisymmetric, reflexive, and transitive relations to the noncommutative setting. Building on these concepts, Weaver quantized posets [13] and showed that *quantum graphs* [2], which are used for quantum error correction, can be understood in terms of quantum relations [14].

Von Neumann algebras are noncommutative generalizations of measure spaces rather than of sets. Kornell identified *hereditarily atomic* von Neumann algebras, which are essentially (possibly infinite) sums of matrix algebras, as the proper noncommutative generalizations of sets [8]. For this reason, hereditarily atomic von Neumann algebras are also called *quantum sets*, and the category **qRel** of quantum sets and quantum relations can be regarded as the proper noncommutative generalization of the category **Rel** of sets and binary relations. Just like **Rel**, but in contrast to the category of all von Neumann algebras and quantum relations, **qRel** is dagger compact closed. Together with Kornell and Mislove, the second author investigated the categorical properties of quantum posets in this restricted setting of hereditarily atomic von Neumann algebras [11]. Building on this work, they introduced *quantum cpos*, which are noncommutative versions of ω-complete partial orders (cpos). Ordinary cpos can be used to construct denotational models of ordinary programming languages, and in a similar way, they showed that quantum cpos can be used for the denotational semantics of quantum programming languages [10]. Also building on the definition of quantum posets in the hereditarily atomic setting, both authors introduced *quantum suplattices* [7], which are noncommutative versions of complete lattices and supremum-preserving maps. For quantum suplattices, the compact structure of **qRel** seems to be essential.

Categorically, quantization via quantum relations can be understood as the internalization of mathematical structures in the category **qRel**, and many theorems about quantized structures via quantum relations rely on the categorical properties of **qRel**. There are several categorical generalizations of the category **Rel** such as allegories [3] or bicategories of relations [1], but unfortunately, **qRel** is not an example of either of them. This is mainly due to the fact that the internal functions of **qRel** form a semicartesian monoidal category rather than a cartesian monoidal category, which reflects the quantum character of **qRel**. Tweaking the definitions of either allegories or bicategories of relations is difficult; their cartesian character seems to be essential, and it cannot be adjusted without tearing down the whole building.

*Supported by VEGA-2/0128/24, VEGA-1/0036/23 and APVV-20-0069
†Supported by VEGA 2/0128/24 and APVV-22-0570

Therefore, we aim to find a different categorical generalization of **Rel** that would capture **qRel**, but also other generalizations of **Rel**, such as the category V-**Rel** of sets and relations with values in a unital commutative quantale V, which is used in fuzzy mathematics [6]. Only if V is a frame, V-**Rel** seems to be a bicategory of relations. We take daggers as a primitive notion, and identify six properties of **qRel** as axioms for our categorical generalization of **Rel**. Similar properties also occur in recent categorical axiomatizations of several dagger categories such as the category **Hilb** and **Rel** [4, 9, 5], and likely will form a subset of the axioms of a future categorical characterization of **qRel**. Hence, we define a *semicartesian category of relations* to be a category **R** such that

(1) **R** is a locally small dagger compact category;

(2) **R** has all small dagger biproducts;

(3) **R** has precisely two scalars;

(4) **R** is a dagger kernel category;

(5) For each object X in **R** there is precisely one morphism $X \to I$ with zero kernel;

(6) For each object X and each projection p on X, $p \geq \mathrm{id}_X$ if and only if $\ker p = 0$.

Here, a *projection* on an object X is a morphism $p : X \to X$ such that $p \circ p = p = p^\dagger$. For the last axiom, we use that the first three axioms imply that **R** is a *quantaloid*, i.e., a category enriched over the category **Sup** of complete lattices and supremum-preserving maps. As another consequence of the axioms, we prove that the homsets of **R** are actually orthomodular lattices. We conclude with a discussion of conditions that assure the existence of a power set construction in semicartesian categories of relations.

References

[1] A. Carboni and R.F.C. Walters. Cartesian bicategories I. *J. Pure Appl. Alg.*, 49(1):11–32, 1987.

[2] R. Duan, S. Severini, and A. Winter. Zero-error communication via quantum channels, noncommutative graphs, and a quantum lovász number. *IEEE Trans. Inf. Theory*, 59:1164–1174, 2013.

[3] P. Freyd and A. Scedrov. *Categories, Allegories*. North-Holland, 1990.

[4] C. Heunen and A. Kornell. Axioms for the category of Hilbert spaces. *Proceedings of the National Academy of Sciences*, 119(9):e2117024119, 2022.

[5] C. Heunen, A. Kornell, and N. van der Schaaf. Axioms for the category of Hilbert spaces and linear contractions. *To appear in Bulletin of the London Mathematical Society*, 2024.

[6] D. Hofmann, G.J. Seal, and W. Tholen. *Monoidal Topology*. Cambridge University Press, 2014.

[7] G. Jenča and B. Lindenhovius. Quantum Suplattices. *Proceedings QPL*, 2023.

[8] A. Kornell. Quantum sets. *J. Math. Phys.*, 61, 2020.

[9] A. Kornell. Axioms for the category of sets and relations. *arXiv:2302.14153*, 2023.

[10] A. Kornell, B. Lindenhovius, and M. Mislove. Quantum CPOs. *Proceedings QPL*, 2021.

[11] A. Kornell, B. Lindenhovius, and M. Mislove. A category of quantum posets. *Indagationes Mathematicae*, 33:1137–1171, 2022.

[12] G. Kuperberg and N. Weaver. *A Von Neumann Algebra Approach to Quantum Metrics: Quantum Relations*. Memoirs of the American Mathematical Society. American Mathematical Society, 2012.

[13] N. Weaver. Quantum relations. *Mem. Amer. Math. Soc.*, 215, 2012.

[14] N. Weaver. Quantum graphs as quantum relations. *The Journal of Geometric Analysis*, 31:9090—-9112, 2021.