On Monadic De Morgan Monoids*

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Monadic Boolean algebras were first systematically studied by Halmos [3] and generalized to Polyadic Boolean algebra [2] which serves as a framework for algebrization of predicate logic. One of the important theorems is that that every monadic Boolean algebra is a subdirect union of functional monadic Boolean algebras. On the other hands, the structure of De Morgan monoids (DMM) have been extensively studied in [4, 5] and its connection with relevance logic [1]. In [6], Andrew Tedder proposed an algebraic framework for Mares-Goldblatt semantics for quantified relevance logics by using as an example De Morgan monoid with a Mares-Goldblatt style interpretation of the quantifiers to study quantified relevance logic. This sheds a light on generalization of monadic De Morgan monoids and polyadic De Morgan monoids as an algebraization of quantified relevance logic. In this talk, we will report our work in progress in this direction of study.

We start with the definition of functional monadic DMM.

Definition 1 (Functional Monadic De Morgan Monoid)**.** *If $\mathfrak{B}$ is a DMM, X is a non-empty set, then the structure $\langle \mathfrak{A}; \wedge, \vee, \sim, \circ, \to, 1\rangle$ is a $\mathfrak{B}$-valued functional monadic DMM, if the following conditions hold :*

1. $\mathfrak{A} \subseteq \mathfrak{B}^X$;
2. *$\mathfrak{A}$ is closed under the 'lifted' operations $\wedge, \vee, \sim, \circ, \to$, and contains 1, where*
   (a) $1(x) =^{\mathfrak{B}} 1$, *for all* $x \in X$;
   (b) $(\sim p)(x) =^{\mathfrak{B}} \sim (p(x))$, *for* $p \in \mathfrak{A}$;
   (c) $(p \otimes q)(x) =^{\mathfrak{B}} p(x) \otimes q(x)$, *for* $p, q \in \mathfrak{A}$ *and* $\otimes \in \{\wedge, \vee, \sim, \circ, \to\}$
3. *The constant function $\forall p$ exists in $\mathfrak{A}$, for each $p \in \mathfrak{A}$, and hence the appropriate generalized meets and joins exists in $\mathfrak{B}$, where we define:*

$$R(p) =_{df} \{p(x) : x \in X\}$$

$$\forall p(x) =_{df} \bigwedge^{\mathfrak{B}} R(p)$$

Then we demonstrate the following property of functional monadic DMM.

Lemma 1. *A functional universal quantifier $\forall$ on a functional monadic DMM satisfies the following:*

(Q_1) $\forall 1 = 1$

(Q_2) $\forall p \leq p$

(Q_3) $\forall(p \wedge q) = \forall p \wedge \forall q$

(Q_4) $\forall\forall p = \forall p = \neg\forall\neg\forall p$

(Q_5) $\forall(p \to q) \leq (\forall p \to \forall q)$

(Q_6) $\forall(\forall p \to \forall q) = \forall p \to \forall q$

(Q_7) $\forall(p \vee q) \leq \neg\forall\neg p \vee \forall q$

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On the other hand, we can define a universal quantifier to be a mapping satisfying certain conditions as follows :

Definition 2 (Universal Quantifier)**.** $\mathfrak{A}$ *be an DMM. A* (Universal) Quantifier *is a map* $\forall : \mathfrak{A} \to \mathfrak{A}$ *that satisfies* (Q_1)*–*(Q_7) *(defined in Lemma 1).*

Some important facts about universal quantifiers can be derived :

Lemma 2. *Let* $\mathfrak{A}$ *be an DMM and* $\forall$ *is a quantifier on* $\mathfrak{A}$*. The following properties hold:*

1. $\forall 0 = 0$
2. $p \in \forall(\mathfrak{A})$ *iff* $\forall p = p$.
3. *If* $\forall p \leq q$ *then* $\forall p \leq \forall q$.
4. *If* $p \leq q$ *then* $\forall p \leq \forall q$.
5. $\forall(p \to q) \leq (\neg\forall\neg p \to \neg\forall\neg q)$
6. $p \leq \forall\neg\forall\neg p$
7. $\forall(\forall p \to q) \leq (\forall p \to \forall q)$

Definition 3. *A Monadic DMM is a tuple* $\langle \mathfrak{A}, \forall \rangle$ *where* $\mathfrak{A}$ *is an DMM, and* $\forall$ *is a quantifier on* $\mathfrak{A}$*.*

In the end of this talk, we will briefly address the problem of two Representation theorems in monadic De Morgan monoid either in terms of functional monadic DMM or in terms of subdirectly irreducible monadic DMM.

References

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