Raney extensions of frames as pointfree T_0 spaces

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Raney duality, as illustrated in [3], consists of a dual equivalence of categories between Raney algebras and T_0 spaces. The dual equivalence sends a space X to the embedding $\Omega(X) \subseteq \mathcal{U}(X)$, where $\Omega(X)$ is its lattice of opens and $\mathcal{U}(X)$ its lattice of saturated sets (the upper sets in the specialization order). In this setting, all Raney algebras are, so to speak, spatial: there are no Raney algebras which are not $\Omega(X) \subseteq \mathcal{U}(X)$ for some space X. We propose to extend Raney duality by extending the category of Raney algebras to a more pointfree category. We consider *Raney extensions*, as introduced in [7], pairs (L, C) where C is a coframe, and $L \subseteq C$ is a frame that meet-generates C such that the inclusion preserves the frame operations as well as the strongly exact meets. Raney extensions are the objects of the category **Raney**, whose morphisms are coframe maps which restrict to frame maps on the first components. We have the following.

Proposition 1. *There is an adjunction* $\Omega_R : \mathbf{Top} \leftrightarrows \mathbf{Raney}^{op} : \mathsf{pt}_R$*, where* $\Omega_R(X) = (\Omega(X), \mathcal{U}(X))$ *for all spaces* X*. In* **Top***, the fixpoints are the* T_0 *spaces.*

We will see that the opposite of the frame $\mathsf{Filt}_{\mathcal{SE}}(L)$ of strongly exact filters and the opposite of the frame $\mathsf{Filt}_{\mathcal{E}}(L)$ of exact filters studied in [6] and [5] are, respectively, the largest and the smallest Raney extension of some frame L. These two frames are known to be, respectively, anti-isomorphic to the collection $\mathsf{S}_{\mathfrak{o}}(L)$ of fitted sublocales and isomorphic to the collection $\mathsf{S}_{\mathfrak{c}}(L)$ of joins of closed sublocales. In [7] the following is shown.

Theorem 2. *For a frame* L*, the largest Raney extension on it is* $(L, \mathsf{S}_{\mathfrak{o}}(L))$*, and the smallest one is* $(L, \mathsf{S}_{\mathfrak{c}}(L)^{op})$*.*

A topological space X is T_D if for all $x \in X$ there are opens $U, V \subseteq X$ such that $\{x\} = U \backslash V$. This axiom is introduced in [1]. In [2], it is shown that the axiom is in a certain sense dual to sobriety, in fact the following is shown.

- A space X is sober if and only if whenever a subspace inclusion $X \subseteq Y$ induces a frame isomorphism $\Omega(X) \cong \Omega(Y)$, then that inclusion is the identity.
- A space X is T_D if and only if whenever a subspace inclusion $Y \subseteq X$ induces a frame isomorphism $\Omega(Y) \cong \Omega(X)$, then that inclusion is the identity.

In [2], the definition of the T_D spectrum $\mathsf{pt}_D(L)$ of a frame L is given. With the following result, we find another sense in which sobriety and the T_D property are dual of one another.

Proposition 3. *For a frame* L*, the spectrum of the smallest Raney extension* $(L, \mathsf{S}_{\mathfrak{c}}(L)^{op})$ *is its* T_D *spectrum* $\mathsf{pt}_D(L)$*. The spectrum of the largest one* $(L, \mathsf{S}_{\mathfrak{o}}(L))$ *is the classical spectrum* $\mathsf{pt}(L)$*. Furthermore, for any Raney extension* (L, C) *we have subspace inclusions* $\mathsf{pt}_D(L) \subseteq \mathsf{pt}_R(L, C) \subseteq \mathsf{pt}(L)$*, up to isomorphism.*

In the context of Raney extensions, unlike that of frames, we have a natural version of the T_1 axiom. Because a space X is T_1 if and only if all subsets are saturated, that is, $\mathcal{U}(X) = \mathcal{P}(X)$, we define a Raney extension (L, C) to be T_1 if C is a Boolean algebra. This enables us to find a characterization of subfitness for frames as the weakest possible version of the T_1 axiom.

Theorem 4. *For a frame L, the following are equivalent.*

- *The frame L is subfit.*
- *The frame L admits a T_1 Raney extension.*
- *The frame L admits a unique T_1 Raney extension.*
- *The Raney extension $(L, \mathsf{S}_{\mathfrak{c}}(L))$ is T_1.*

We will see that Raney extensions generalize canonical extensions for distributive lattices, and for locally compact frames (see [4]). We say that an element $c \in C$ for a Raney extension (L, C) is *compact* if it is inaccessible by directed joins of families in L. We say that a Raney extension is *algebraic* when it is generated by its compact elements. We have the following ([7]).

Proposition 5. *For a pre-spatial frame L, the canonical extension (L, L^{δ}) is the free algebraic Raney extension over it.*

References

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