An extension of Stone duality to T_0-spaces and sobrifications

Liping Zhang[1], Xiangnan Zhou[2] and Qingguo Li[3]

School of Mathematics, Hunan University, Changsha, Hunan, 410082, China
[1]lipingzhang@hnu.edu.cn; [2]xnzhou@hnu.edu.cn; [3]liqingguoli@aliyun.com

De Vries [5] developed a renowned de Vries duality between the category **KHaus** of compact Hausdorff spaces with continuous maps and the category **Dev** of de Vries algebras with de Vries morphisms. In [1], Bezhanishvili and Harding extended de Vries duality to stably compact spaces by replacing the category of de Vries algebras with regular proximity frames. They established a de Vries duality between the category **StKSp** of stably compact spaces with proper maps and the category **RPrFrm** of regular proximity frames with proximity morphisms.

In [7], Smyth generalized the compactifications of completely regular spaces to the stable compactifications of T_0-spaces. He showed that the equivalence classes of stable compactifications of a given T_0-space form a poset. The largest element, named as the *Smyth compactification* in [3], is a generalization of the Stone-Čech compactification.

It is well known that the Stone-Čech compactification yields a reflector $\beta : \mathbf{CReg} \to \mathbf{KHaus}$ between the category **CReg** of completely regular spaces with continuous maps and the category **KHaus**. That is, **KHaus** is a full reflective subcategory of **CReg**. In [4], by introducing the category **Comp** of compactifications of completely regular spaces, Bezhanishvili, Morandi and Olberding proved that the category **CReg** is equivalent to the full subcategory **SComp** of **Comp** consisting of Stone-Čech compactifications of completely regular spaces. To develop the de Vries duality for completely regular spaces, they introduced the category **DeVe** of de Vries extensions, and built the dual equivalence between the categories **Comp** and **DeVe**. Under this duality, the full subcategory **MDeVe** of **DeVe** comprising maximal de Vries extensions was placed into duality with the category **SComp** of Stone-Čech compactifications of completely regular spaces.

Bezhanishvili and Harding developed two methods to establish the duality for T_0-spaces. On the one hand, in [3], by considering the category **StComp** of stable compactifications of T_0-spaces, they proved that the full subcategory **Smyth** of **StComp** composed of smyth compactifications of T_0-spaces is equivalent to the category $\mathbf{Top_0}$ of T_0-spaces. To extended the de Vries duality of stably compact spaces to T_0-spaces, they introduced the category **RE** of Raney extensions and established a duality between the categories **StComp** and **RE**. Thus it yielded a duality between the category $\mathbf{Top_0}$ and the full subcategory **MRE** of **RE** consisting of maximal Raney extensions. On the other hand, in [2], they developed an alternate duality between the category $\mathbf{Top_0}$ and the category **RAlg** of Raney algebras.

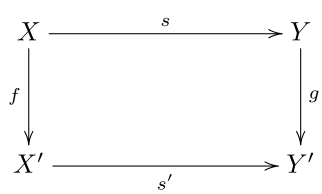
As we all know, sober spaces are closely related to pointfree topology and logic because of the duality (*Kawahara duality*) for spatial frames (see [6]). And the category **Sob** of sober spaces is a full reflective subcategory of the category $\mathbf{Top_0}$. In this paper, instead of stably compactifications of T_0-spaces, we choose to employ sobrifications of T_0-spaces to construct a new duality for T_0-spaces. We introduce the definition of a spatial frame Raney extension as follows.

Definition 1. Let L be a spatial frame and K a Raney lattice, where Raney lattice is a completely distributive complete lattice generated by completely join-irreducible elements. A frame homomorphism $\varepsilon : L \to K$ is said to be a *spatial frame Raney extension* if it is injective and $\varepsilon(L)$ is dense in K.

Then we establish a one-to-one correspondence between spatial frame Raney extensions and sobrifications of T_0-spaces. In order to build a duality for T_0-spaces, we introduce the category of spatial frame Raney extensions and the category of sobrifications of T_0-spaces as follows.

Definition 2. The category of spatial frame Raney extensions, denoted by **SFrmRE**, is the category whose objects are spatial frame Raney extensions $\varepsilon : L \to K$ and whose morphisms are pairs (ϕ, ψ) where $\phi : L \to L'$ is a frame homomorphism, $\psi : K \to K'$ is a complete lattice homomorphism, and $\varepsilon' \circ \phi = \psi \circ \varepsilon$.

Definition 3. The category of sobrifications, denoted $\mathbf{Sob_f}$, is the category whose objects are sobrifications $s : X \to Y$ and whose morphisms are pairs (f, g) of continuous maps, and the following diagram commutes:



We obtain one of the main theorem of this paper.

Theorem 4. The categories **SFrmRE** and $\mathbf{Sob_f}$ are dually equivalent; and the category $\mathbf{Sob_f}$ is equivalent to the category $\mathbf{Top_0}$.

Therefore, by Theorem 4, we obtain the duality for T_0-spaces.

Theorem 5. The category $\mathbf{Top_0}$ is dually equivalent to the category **SFrmRE**.

Especially, we apply the duality for T_0-spaces to its full subcategory $\mathbf{CKTop_0}$ consisting of core-compact T_0-spaces. And we denote the category **CFrmRE** be the full subcategory of **SFrmRE** consisting of continuous frame Raney extensions, where continuous frame Raney extension is a special spatial frame Raney extension $\varepsilon : L \to K$ with L as a continuous frame. Then we obtain the following result.

Theorem 6. There is a dual equivalence between the categories $\mathbf{CKTop_0}$ and **CFrmRE**.

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