Finite Homomorphism Preservation in Many-Valued Logics

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A canonical result in model theory is the Homomorphism Preservation Theorem (h.p.t.) which states that a first-order formula is preserved under homomorphisms on all structures if and only if it is equivalent to an existential-positive formula. It is an example of a preservation theorem, linking a syntactic class of formulas with preservation under a particular kind of map between structures and standardly proved via a compactness argument. Rossman [1] established that the h.p.t. remains valid when restricted to finite structures, yielding the following formulation of the theorem.

Finite Homomorphism Preservation Theorem A first-order sentence of quantifier-rank n is preserved under homomorphisms on finite structures iff it is equivalent in the finite to an existential-positive sentence of quantifier rank $\rho(n)$ (for some explicit function $\rho : \omega \to \omega$).
That is, for any first-order sentence ϕ of quantifier rank n, $Mod_{fin}(\phi)$ is closed under homomorphisms iff there is an existential-positive sentence ψ of quantifier rank $\rho(n)$ such that for all finite models $M \models \phi$ iff $M \models \psi$.

This is a significant result in the field of finite model theory. It stands in contrast to other results proved via compactness, including the other preservation theorems where the failure of the compactness also results in the failure of the derived theorem [2]. It is also an important result for the field of constraint satisfaction due to the equivalence of existential-positive formulas and unions of conjunctive queries. Adjacently, Dellunde and Vidal [3] established that a version of the h.p.t. holds for a collection of many-valued models, those defined over a fixed finite MTL-chain.

MTL Finite Homomorphism Preservation Theorem Let $\mathcal{P}$ be a predicate language, A a finite MTL-chain and ϕ a consistent sentence over A. Then ϕ is equivalent over A to an existential-positive sentence iff $Mod^{A}_{fin}(\phi)$ is closed under homomorphisms.

MTL-algebras provide the algebraic semantics for the monoidal t-norm logic MTL, a basic propositional fuzzy logic that encompasses the most well-studied fuzzy logics including Hájek's basic logic BL, Gödel–Dummett logic G and Łukasiewicz logic Ł [5, Chapter 1, Section 2]. Much like Rossman's work, Dellunde and Vidal's investigation is further motivated by the application of models defined over MTL-algebras to *valued* constraint satisfaction problems (VCSP), a generalisation of classical CSP, where constraints are assigned some form of weighting which is optimised for in the solution. This has been effectively modelled by taking the weights as elements of an algebra and utilising the algebraic operations to interpret their combination in a potential solution [4], MTL-algebras providing one example. Our investigation picks up at the meeting point of these two strands. One can extend Rossman's proof of a finite h.p.t. to a very wide collection of many-valued models, which in particular establishes a finite variant to Dellunde and Vidal's result. In fact, we work with more general algebras than MTL-chains, the somewhat artificial class of algebras we refer to as interpreting algebras and we consider the case where we allow our models to be defined over varying interpreting algebras.

Definition An *interpreting algebra* is an algebra A in signature $\mathcal{L} = \langle \wedge, \vee, \&, 1\rangle$ such that:

$\langle A, \wedge, \vee\rangle$ is a distributive lattice; $\langle A, \&, 1\rangle$ is a commutative (abelian) monoid;

$\forall a, b, c \in A, a \leq b$ implies $a \;\&\; c \leq b \;\&\; c$. $\forall a, b \in A, a \vee b \geq 1$ implies $a \geq 1$ or $b \geq 1$.

In the many-valued setting both the notion of homomorphism and existential-positive formulas split into a number of interrelated concepts and this naturally provides a number of possible generalisations of the classical h.p.t. As it turns out, the appropriate variant links *protomorphisms* with *existential-$\wedge$-positive sentences* ($\exists.\wedge$.p).

Definition Let $(A, M), (B, N)$ be $\mathcal{P}$-models. A map $g\colon M \to N$ is a *protomorphism* from (A, M) to (B, N) iff:

- for every $F \in \mathcal{P}$ and $\bar{m} \in M$ $g(F^M(\bar{m})) = F^N(g(\bar{m}))$.
- for every $R \in \mathcal{P}$ and $\bar{m} \in M$ $R^M(\bar{m}) \geq 1$ implies $R^N(g(\bar{m})) \geq 1$.

Let $f\colon A \to B$ and $g\colon M \to N$ be maps. We call the pair $(f, g)\colon (A, M) \to (B, M)$ a *homomorphism* from (A, M) to (B, N) iff f is an algebraic $\mathcal{L}$-homomorphism and g is a protomorphism from (A, M) to (B, N). We write $\to_p$ $(\to)$ to indicate there exists a protomorphism (homomorphism) between two $\mathcal{P}$-models.

Given a predicate language $\mathcal{P}$ and a $\mathcal{P}$-formula ϕ it is said that ϕ is existential-$\wedge$-positive iff ϕ is built using the connectives $\wedge$ and $\vee$ and the existential quantifier $\exists$.

One can easily check by induction that $\exists.\wedge$.p sentences are preserved under protomorphisms. Our strategy for the other direction is to translate between $\mathcal{P}$-models defined over interpreting algebras and a 'classical counterpart' in such a way that the behaviour regarding protomorphisms and $\exists.\wedge$.p-sentences is preserved. The classical translations are presented as a $\mathcal{P}$-model taken over the 2 element Boolean algebra $\{\top, \bot\}$.

Definition Let (A, M) be a $\mathcal{P}$-model over an interpreting algebra A. We define the $\mathcal{P}$-model $(\{\top, \bot\}, M^\top)$, also denoted simply as $M^\top$ as follows:

$$R^{M^\top}(\bar{m}) = \begin{cases} \top & \text{if } R^M(\bar{m}) \geq 1_A \\ \bot & \text{if } R^M(\bar{m}) < 1_A. \end{cases}$$

One can then apply Rossman's results to these objects (viewed as a classical models) before pulling back into the many-valued setting, yielding our many-valued equivalent.

Finite Protomorphism Preservation Theorem Let $\mathcal{P}$ be a predicate language and ϕ a consistent $\mathcal{P}$-sentence. Then ϕ is equivalent in the finite to an $\exists.\wedge$.p-sentence ψ iff $Mod_{fin}(\phi)$ is preserved under protomorphisms.

Moreover, when one restricts to models defined over a fixed algebra, the usual notion of homomorphism collapses with protomorphisms. This lets us freely add it to the equivalence.

Fixed Finite Homomorphism Preservation Theorem Let $\mathcal{P}$ be a predicate language, A an interpreting algebra and ϕ a consistent $\mathcal{P}$ sentence over A in the finite. The following are equivalent:

1. ϕ is equivalent over A in the finite to an $\exists.\wedge$.p sentence ψ, i.e. there is an $\exists.\wedge$.p-sentence $\psi : Mod^A_{fin}(\phi) = Mod^A_{fin}(\psi)$.
2. ϕ is preserved under protomorphisms on A, i.e. $Mod^A_{fin}(\phi)$ is closed under $\to_p$.
3. ϕ is preserved under homomorphisms on A, i.e. $Mod^A_{fin}(\phi)$ is closed under $\to$.

References

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