Finite lattices, N-free posets and orthomodularity

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Dacey orthosets

In his PhD. thesis [1], Dacey explored the notion of "abstract orthogonality" through sets equipped with a symmetric, irreflexive relation $\perp$. He named these structures *orthogonality spaces*. More recently, these structures have been referred to as *orthosets* [6, 5] and we will adopt this terminology in this text. Note that an orthoset is just a simple graph.

Definition 1. An *orthoset* $(O, \perp)$ is a set O equipped with an irreflexive symmetric binary relation $\perp \subseteq O \times O$, called *orthogonality*.

For every subset X of an orthoset O, we write

$$X^{\perp} = \{y \in O \mid \text{for all } x \in X,\, x \perp y\}$$

Graph-theoretically, this is just the set of all common neighbours of a set of vertices. It is easy to see that $X \mapsto X^{\perp\perp}$ is a closure operator; a subset of an orthoset $(O, \perp)$ with $X = X^{\perp\perp}$ is called *orthoclosed*. The orthoclosed subsets of an orthoset O form a complete ortholattice $L(O, \perp)$, which we call *the logic of* $(O, \perp)$.

Arguably the most significant theorem established in Dacey's thesis [1] is the following.

Theorem 1. *Let $(O, \perp)$ be an orthoset. Then $L(O, \perp)$ is an orthomodular lattice if and only if for every orthoclosed subset X and every maximal pairwise orthogonal set (or a clique) $B \subseteq X$, $B^{\perp\perp} = X^{\perp\perp}$.*

The orthosets that meet one of the equivalent conditions of Theorem 1 are called *Dacey orthosets*.

The basic idea

Recently, we have achieved moderate success using the following straightforward approach.

1. We consider some class of (finite) objects $\mathcal{X}$.
2. For every object $X \in \mathcal{X}$, we construct in some way an orthoset $\mathcal{O}(X)$.
3. We characterize those objects $X \in \mathcal{X}$ for which $\mathcal{O}(X)$ is a Dacey orthoset.
4. We characterize those objects $X \in \mathcal{X}$ for which $\mathcal{O}(X)$ is a Dacey orthoset with a Boolean logic.

In the talk, we will present several results we found using this approach. The class $\mathcal{X}$ will be always some class of posets. We will consider two types of a construction of an orthoset from a poset: the orthoset of quotients and the incomparability orthoset.

Orthosets of quotients

For a poset P, we write $Q^{+}(P)$ for the set of all pairs $(a, b) \in P \times P$ with $a < b$. In lattice theory, the elements of $Q^{+}(A)$ are called *proper quotients*. An element $(a, b) \in Q^{+}(P)$ is denoted by $[a < b]$.

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For $[a < b], [c < d] \in Q^+(P)$ we write $[a < b] \perp [c < d]$ if $b \leq c$ or $d \leq a$. Clearly, $\perp$ is symmetric and irreflexive, so $(Q^+(P), \perp)$ is an orthoset.

Theorem 2. *[3] Let P be a finite bounded poset. Then P is a lattice if and only if its orthoset of quotients $(Q^+(P), \perp)$ is Dacey.*

Theorem 3. *[3] Let P be a bounded poset. Then P is a chain if and only if $L(Q^+(P), \perp)$ is a Boolean algebra.*

Incomparability orthosets

Let P be a poset. For $x, y \in P$ let us now write $x \perp y$ if and only if x, y are incomparable. We say that $(P, \perp)$ is the *incomparability orthoset* of P.

Let P be a poset. For a quadruple of elements $(a, b, c, d) \in P^4$, we say that they *form an N* if and only if $a < c \succ b < d$ (note the covering relation here), $b < d$, and all the other distinct pairs of elements of the set $\{a, b, c, d\}$ are incomparable. We denote this by $N(a, b, c, d)$. A poset such that no quadruple of elements forms an N is called *N-free*.

N-free posets were introduced by Grillet in [2]. In that paper, the following characterization of N-free posets was proved.

Theorem 4. *A finite poset P is N-free if and only if every maximal chain in P intersects every maximal antichain in P.*

Theorem 5. *[4] Let P be a finite poset. Then P is N-free if and only if its incomparability orthoset $(P, \perp)$ is Dacey.*

We further characterize the finite posets P with a Boolean $L(P, \perp)$ by the absence a more general type of small substructure, which we term a *weak N*; a weak N is like the N defined earlier, with the distinction that we allow a and d to be comparable.

Theorem 6. *[4] Let P be a finite poset. Then $L(P, \perp)$ is Boolean iff there is no weak N in P.*

References

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