Varieties of MV-monoids and positive MV-algebras

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We investigate MV-monoids and their subvarieties. An *MV-monoid* is an algebra $\langle A, \vee, \wedge, \oplus, \odot, 0, 1\rangle$ where:

- $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice;
- $\langle A, \oplus, 0\rangle$ and $\langle A, \odot, 1\rangle$ are commutative monoids;
- $\oplus$ and $\odot$ distribute over $\vee$ and $\wedge$;
- for every $x, y, z \in A$,

$$(x \odot y) \oplus ((x \oplus y) \odot z) = (x \oplus (y \odot z)) \odot (y \oplus z);$$
$$(x \odot y) \oplus z = ((x \oplus y) \odot ((x \odot y) \oplus z)) \vee z;$$
$$(x \oplus y) \odot z = ((x \odot y) \oplus ((x \oplus y) \odot z)) \wedge z.$$

Every MV-algebra in the signature $\{\oplus, \neg, 0\}$ is term equivalent to an algebra that has an MV-monoid as a reduct, by defining, as standard, $1 \coloneqq \neg 0$, $x \odot y \coloneqq \neg(\neg x \oplus \neg y)$, $x \vee y \coloneqq (x \odot \neg y) \oplus y$ and $x \wedge y \coloneqq \neg(\neg x \vee \neg y)$. We show that every subdirectly irreducible MV-monoid $\mathbf{A}$ is totally ordered and satisfies the property: for all $x, y \in A$, $x \oplus y = 1$ or $x \odot y = 0$.

Using this result, we investigate the bottom part of the lattice of subvarieties of MV-monoids, characterizing all the almost minimal varieties of MV-monoids as the varieties generated by:

- a reduct of a finite MV-chain of prime order ($\mathbf{Ł}_p^+$);
- the unique MV-monoid $\mathbf{C}_2^{\Delta}$ on the 3-element chain $0 < \varepsilon < 1$ satisfying $\varepsilon \oplus \varepsilon = \varepsilon$ and $\varepsilon \odot \varepsilon = 0$;
- the dual of $\mathbf{C}_2^{\Delta}$.

One of the main tools that we used to develop the theory of MV-monoids is the categorical equivalence Γ between unital commutative ℓ-monoids and MV-monoids [1].

A *unital commutative ℓ-monoid* is an algebra $\langle M, \vee, \wedge, +, 1, 0, -1\rangle$ with the following properties:

- $\langle M, \vee, \wedge, +, 0\rangle$ is a commutative ℓ-monoid;
- $-1 + 1 = 0$;
- $-1 \leq 0 \leq 1$;

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- for all $x \in M$ there is $n \in \mathbb{N}$ such that

$$\underbrace{(-1)+\cdots+(-1)}_{n \text{ times}} \leq x \leq \underbrace{1+\cdots+1}_{n \text{ times}}.$$

Thus the relationship between unital commutative ℓ-monoids and MV-monoids is similar to the one between unital abelian ℓ-groups and MV-algebras and we exploit this fact in several statements of our work. We also present a version of Hölder's theorem for unital commutative ℓ-monoids.

Particular examples of MV-monoids are positive MV-algebras, i.e. the $\{\vee, \wedge, \oplus, \odot, 0, 1\}$-subreducts of MV-algebras or, equivalently, the proper subquasivariety of the variety of MV-monoids (MVM), axiomatized relatively to MVM by

$$(x \oplus z \approx y \oplus z \text{ and } x \odot z \approx y \odot z) \implies x \approx y.$$

Positive MV-algebras form a peculiar quasivariety in the sense that, albeit having a logical motivation (being the quasivariety of subreducts of MV-algebras), it is not the equivalent quasivariety semantics of any logic in the sense of [2].

In this cancellative setting, we characterized the varieties of positive MV-algebras as precisely the varieties generated by finitely many reducts of finite nontrivial MV-chains. We also proved that such reducts coincide with the subdirectly irreducible finite positive MV-algebras. Using these results we prove that: a variety of positive MV-algebras is of the form $\mathcal{V}(\mathcal{K}_I)$, where I is a finite subset of $\mathbb{N} \setminus \{0\}$ containing all the divisors of its elements (*divisor-closed subset*) and $\mathcal{K}_I$ is the set of all reducts of MV-chains $\mathbf{Ł}_m^+$ such that $|Ł_m^+| - 1 \in I$.

In conclusion, we present axiomatizations of all the varieties of positive MV-algebras, using a strategy similar to the one of Di Nola and Lettieri [3]. To do so we define the following set of equations.

Let $I \subseteq \mathbb{N}$ be a divisor-closed set, and let m be the maximum of I (with the convention that $m = 0$ if $I = \emptyset$). We define Σ_I as the set of equations given by:

$$(m+1)x \approx mx \qquad \text{and} \qquad m((k-1)x)^k \approx (kx)^m \tag{1}$$

for all $1 \leq k \leq m$ such that $k \notin I$. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ we define the unary term $\tau_{n,k}(x)$ inductively on n as follows:

$$\tau_{0,k}(x) \coloneqq \begin{cases} 1 & \text{if } k \leq -1, \\ 0 & \text{if } k \geq 0, \end{cases} \qquad \tau_{n+1,k}(x) = \tau_{n,k-1}(x) \odot (x \oplus \tau_{n,k}(x)).$$

For every $n \in \mathbb{N}$, let Φ_n be the following set of equations, for k ranging in $\{0, \ldots, n-1\}$:

$$\tau_{n,k}(x) \oplus \tau_{n,k}(x) \approx \tau_{n,k}(x) \text{ and } \tau_{n,k}(x) \odot \tau_{n,k}(x) \approx \tau_{n,k}(x). \tag{2}$$

Theorem. Let I be a divisor-closed finite set; then $\mathcal{V}(\mathcal{K}_I)$ is axiomatized by $\Phi_{\mathrm{lcm}(I)} \cup \Sigma_I$ relatively to the variety of MV-monoids.

References

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