Induced congruences in σ-frames

Raquel Bernardes

CMUC, University of Coimbra, Portugal
rbernardes@mat.uc.pt

Just as frames generalise topological spaces ([3]), σ-frames generalise σ-topological spaces and, consequently, measurable spaces. Let us recall that a *σ-frame* [1] is a join-σ-complete lattice (that is, a lattice with countable joins) satisfying the distributive law

$$\Big(\bigvee_{a\in A} a\Big) \wedge b = \bigvee_{a\in A} (a \wedge b)$$

for every countable $A \subseteq L$ and $b \in L$. A map between σ-frames is called a *σ-frame homomorphism* if it preserves finite meets (including the top element 1 given by empty meet) and countable joins (including the bottom element 0 given by empty join).

Following Simpson [4], where a new approach to the problem of measuring subsets was proposed, we have been interested in approaching measure theory in the category of σ-frames and σ-frame homomorphisms. In a follow-up to our study of measurable functions in [2], we intend to investigate whether the notion of σ-sublocale generalises the notion of σ-subspace in a way similar to the case of sublocales versus subspaces. There is a difficulty that we will face: contrarily to what happens in the pointfree setting of frames and locales, where sublocales of a given locale L have a concrete description as subsets of L ([3]), the subobjects in the category of σ-locales and σ-localic maps (that is, the dual category of the category of σ-frames and σ-frame homomorphisms), can only be described as *σ-frame congruences* θ on L, that is, equivalence relations on L satisfying the congruence properties

$$(x, y), (x', y') \in \theta \;\Rightarrow\; (x \wedge x', y \wedge y') \in \theta,$$

$$(x_a, y_a) \in \theta \;(a \in A, A = \text{countable set}) \;\Rightarrow\; \Big(\bigvee_{a\in A} x_a, \bigvee_{a\in A} y_a\Big) \in \theta.$$

This is a remarkable difference between the categories of σ-locales and locales.

In this talk, given a σ-space X (that is, a set X equipped with a collection of open sets $\mathcal{O}(X) \subseteq \mathcal{P}(X)$ closed under finite meets and countable joins) and its σ-complete lattice of open sets $\mathcal{O}(X)$, we will focus on the congruences on the σ-frame $\mathcal{O}(X)$ that represent the σ-subspaces of X, referred to as *induced congruences*. We will show that when X is a T_D σ-space, there is a bijection between the σ-subspaces of X and the congruences induced by them.

Let X be a σ-space. From the well-known dual adjunction between the category of σ-spaces and σ-continuous maps and the category of σ-frames and σ-frame homomorphisms,

$$\sigma\mathsf{Top} \underset{\mathsf{Pt}}{\overset{\mathcal{O}}{\underset{\perp}{\rightleftarrows}}} \sigma\mathsf{Frm}\ ,$$

one sees that given a σ-subspace $Y \subseteq X$ (with the induced subspace σ-topology) and the inclusion $j\colon Y \hookrightarrow X$, the congruence

$$\theta_{S_Y} := \{(U, V) \in \mathcal{O}(X) \times \mathcal{O}(X) \mid U \cap Y = V \cap Y\}$$

represents an isomorphic imprint of $\mathcal{O}(Y)$ in $\mathcal{O}(X)$. We call it the *congruence induced by the σ-subspace Y.*

We say that a σ-space satisfies the axiom T_D if for any $x \in X$, there is $U_x \in \mathcal{O}(X)$ such that $x \in U_x$ and $U_x \setminus \{x\}$ is still in $\mathcal{O}(X)$. We show that the representation

$$\pi\colon Y \mapsto \theta_Y$$

is one-to-one whenever X is a T_D σ-space:

Proposition. *For a σ-space X, the map $\pi\colon \mathcal{P}(X) \to \mathcal{C}(\mathcal{O}(X))$ from the powerset of X to the congruence lattice of $\mathcal{O}(X)$ is one-to-one if and only if X is T_D. Moreover, it takes arbitrary joins to arbitrary meets but not finite meets to finite joins.*

We will conclude, moreover, still under axiom T_D, that

$$\pi(\mathcal{P}(X)) \subseteq \mathcal{C}_b(\mathcal{O}(X)),$$

where $\mathcal{C}_b(\mathcal{O}(X))$ denotes the subset of $\mathcal{C}(\mathcal{O}(X))$ consisting of all meets of complemented congruences.

We will finish with the remark that imposing a σ-space to be T_D is not too strong, as it encompasses most of the measurable spaces of importance in measure theory, such as euclidean spaces $\mathbb{R}^n$, separable metric spaces or any T_1-space with a countable basis.

References

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