Kites and pseudo MV-algebras

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We deal with variants of a special construction of certain algebras called *kites*, naturally associated with a noncommutative generalisation of BL-algebras known as *pseudo BL-algebras*. We focus on *perfect pseudo MV-algebras* generalising previous results by DiNola, Dvurečenskij and Tsinakis. The following varieties of algebras will also play an important role: LG – lattice-ordered groups, $\mathsf{CanIGMV}$ – cancellative integral generalised MV-algebras, $\Psi\mathsf{MV}$ – pseudo MV-algebras.

Definition 1. *An $\mathsf{FL_w}$-algebra $\mathbf{A}$ is said to be* perfect *if there is a homomorphism $h_{\mathbf{A}}\colon \mathbf{A} \to \mathbf{2}$ such that for any $x \in h_{\mathbf{A}}^{-1}(0)$ and any $y \in h_{\mathbf{A}}^{-1}(1)$ the inequality $x \leq y$ holds.*

We say that a variety $\mathcal{V}$ of $\mathsf{FL_w}$-algebras is *perfectly generated* if it is generated by its perfect members. If a perfectly generated variety $\mathcal{V}$ is a subvariety of some larger variety, and that larger variety has a well established name, say NN, we write PNN for the perfectly generated subvariety of NN.

Let $\mathbf{A}$ be an FL-algebra, and $a, b \in A$. The *left conjugate* of $a \in A$ by $b \in A$ is the element $\lambda_b(a) := (b \setminus ab) \wedge 1$ and the *right conjugate* is $\rho_b(a) := (ba \,/\, b) \wedge 1$. A *conjugation polynomial* $\boldsymbol{\alpha}$ over $\mathbf{A}$ is any unary polynomial $(\gamma_{a_1} \circ \gamma_{a_2} \circ \cdots \circ \gamma_{a_n})(x)$ where $\gamma \in \{\lambda, \rho\}$ and $a_i \in A$ for $1 \leq i \leq n$. We write $\mathrm{cPol}(\mathbf{A})$ for the set of all conjugation polynomials over $\mathbf{A}$. For an element $u \in A$, an *iterated conjugate* of u is $\boldsymbol{\alpha}(u)$ for some $\boldsymbol{\alpha} \in \mathrm{cPol}(\mathbf{A})$.

Theorem 1. *A subvariety $\mathcal{V}$ of $\mathsf{FL_w}$ is perfectly generated if and only if $\mathcal{V}$ is nontrivial and satisfies the following identities:*

$$\boldsymbol{\alpha}(x \,/\, x^-) \vee \boldsymbol{\beta}(x^- \,/\, x) = 1, \tag{1}$$

$$\boldsymbol{\alpha}((x \vee x^-) \cdot (y \vee y^-))^- \leq \boldsymbol{\alpha}((x \vee x^-) \cdot (y \vee y^-)), \tag{2}$$

$$x \wedge x^- \leq y \vee y^- \tag{3}$$

for every $\mathbf{A} \in \mathcal{V}$ and all $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathrm{cPol}(\mathbf{A})$.

A natural generalisation of MV-algebras is the variety $\Psi\mathsf{MV}$ of *pseudo MV-algebras*. By a result of Dvurečenskij, pseudo MV-algebras are categorically equivalent to the class of lattice-ordered groups with strong unit, so in a good sense they are the largest possible generalisation of MV-algebras for which a Mundici-type categorical equivalence holds. For pseudo MV-algebras, we define a natural generalisation of the kite construction.

Definition 2. *Let $\mathbf{L}$ be an ℓ-group and $\lambda\colon \mathbf{L} \to \mathbf{L}$ be an automorphism. We define the algebra*

$$\mathcal{K}(\mathbf{L}, \lambda) := (L^- \uplus L^+; \wedge, \vee, \odot, \setminus, /, 0, 1)$$

where $L^- \uplus L^+$ is a disjoint union, $0 := e \in L^+$, $1 := e \in L^-$, and the other operations are given by

$$x \wedge y := \begin{cases} x \wedge y & \textit{if } x, y \in L^-, \\ x & \textit{if } x \in L^+, y \in L^- \\ y & \textit{if } x \in L^-, y \in L^+, \\ x \wedge y & \textit{if } x, y \in L^+, \end{cases} \qquad x \vee y := \begin{cases} x \vee y & \textit{if } x, y \in L^-, \\ y & \textit{if } x \in L^+, y \in L^- \\ x & \textit{if } x \in L^-, y \in L^+, \\ x \vee y & \textit{if } x, y \in L^+, \end{cases}$$

$$x \odot y := \begin{cases} x \cdot y & \textit{if } x, y \in L^-, \\ \lambda(x) \cdot y \vee 0 & \textit{if } x \in L^-, y \in L^+ \\ x \cdot y \vee 0 & \textit{if } x \in L^+, y \in L^-, \\ 0 & \textit{if } x, y \in L^+, \end{cases}$$

$$x \setminus y := \begin{cases} x^{-1} \cdot y \wedge 1 & \textit{if } x, y \in L^-, \\ 1 & \textit{if } x \in L^+, y \in L^- \\ \lambda(x)^{-1} \cdot y \vee 0 & \textit{if } x \in L^-, y \in L^+, \\ x^{-1} \cdot y \wedge 1 & \textit{if } x, y \in L^+, \end{cases} \qquad y \,/\, x := \begin{cases} y \cdot x^{-1} \wedge 1 & \textit{if } x, y \in L^-, \\ 1 & \textit{if } x \in L^+, y \in L^- \\ y \cdot x^{-1} \vee 0 & \textit{if } x \in L^-, y \in L^+, \\ \lambda^{-1}(y \cdot x^{-1}) \wedge 1 & \textit{if } x, y \in L^+, \end{cases}$$

Theorem 2. *Let* $\mathbf{A}$ *be a perfect pseudo MV-algebra. Then* $\mathbf{A} \cong \mathcal{K}(\ell(\mathbf{F_A}), \ell^{\approx})$*, where* $\ell^{\approx}$ *is the automorphism induced by the term operation* $x^{\approx} := 0\backslash(0\backslash x)$*.*

The next theorem generalises some results by Di Nola, Dvurečenskij and Tsinakis.

Theorem 3. *The category* $\mathsf{pf\Psi MV}$ *of perfect pseudo MV-algebras is equivalent to the category of lattice-ordered groups with a distinguished automorphism.*

We also obtain a characterisation of varieties generated by kites, and a description of the lattice of such varieties. For a variety $\mathcal{V}$ of algebras, we let $\Lambda(\mathcal{V})$ stand for the lattice of subvarieties of $\mathcal{V}$. If the poset of nontrivial subvarieties of $\mathcal{V}$ is also a lattice, we let $\Lambda^+(\mathcal{V})$ stand for that lattice. We denote by $\mathbb{D}$ the *divisibility lattice*, that is, $\mathbb{N}$ ordered by divisibility. The parameter n and $dim(\mathcal{V})$ below refer to a notion of *dimension* of a variety, which we leave undefined here for lack of space.

Definition 3. *We define two pairs of maps*

$$\psi \colon \Lambda(\mathsf{P\Psi MV}) \to \Lambda(\mathsf{CanlGMV}), \ \textit{where } \psi(\mathcal{V}) = V\{\mathbf{F_A} : \mathbf{A} \in \mathcal{V}_{pf}\},$$
$$\Psi \colon \Lambda(\mathsf{P\Psi MV}) \to \Lambda(\mathsf{CanlGMV}) \times \mathbb{D}, \ \textit{where } \Psi(\mathcal{V}) = (\psi(\mathcal{V}),\, dim(\mathcal{V})),$$

for any $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$ *and*

$$\delta \colon \Lambda(\mathsf{CanlGMV}) \to \Lambda(\mathsf{P\Psi MV}), \ \textit{where } \delta(\mathcal{V}) = V\{\mathbf{A} \in \mathsf{pf\Psi MV} : \mathbf{F_A} \in \mathcal{V}\},$$
$$\Delta \colon \Lambda(\mathsf{CanlGMV}) \times \mathbb{D} \to \Lambda(\mathsf{P\Psi MV}), \ \textit{where } \Delta(\mathcal{V}, n) = \delta(\mathcal{V}) \cap \mathsf{P\Psi MV}_n,$$

for any $\mathcal{V} \in \Lambda(\mathsf{CanlGMV})$ *and* $n \in \mathbb{D}$*.*

Theorem 4. *Let* $\mathcal{V} \in \Lambda(\mathsf{P\Psi MV})$*. The following are equivalent.*

1. $\mathcal{V}$ *is generated by kites.*
2. $\mathcal{V} = \Delta\Psi(\mathcal{V})$*.*
3. $\mathcal{V} = \Delta(\mathcal{W}, n)$ *for some* $\mathcal{W} \in \Lambda(\mathsf{CanlGMV})$ *and some* $n \in \mathbb{D}$*.*

Theorem 5. *Let* $\mathbb{K}$ *be the lattice of subvarieties of* $\mathsf{P\Psi MV}$ *generated by kites.*

$$\mathbb{K} \cong \mathbf{1} \oplus \big(\Lambda^+(\mathsf{CanlGMV}) \times \mathbb{D}\big) \cong \mathbf{1} \oplus \big(\Lambda^+(\mathsf{LG}) \times \mathbb{D}\big)$$

where $\mathbf{1}$ *is the trivial lattice and* $\oplus$ *is the operation of ordinal sum.*