Equivalential Algebras with Regular Semilattice

Eryk Tadeusz Lipka[1]

University of the National Education Comission, Kraków, Polska
eryklipka0@gmail.com

Abstract

Consider two operations on a Heyting algebra: $x\widehat{\wedge}y = \neg\neg x \wedge \neg\neg y, x \cdot y = x \to y \wedge y \to x$ and a class $\mathcal{H}$ of all $(\cdot, \widehat{\wedge})$ subreducts of Heyting algebras. It is easy to see that $\mathcal{H}$ is a quasivariety and all algebras in this class are Fregean (1-regular and congruence orderable) and congruence permutable. We show that $\mathcal{H}$ is actually a variety by characterizing it as an equational class. In the language of Tame Congruence Theory this class is an example of a mixed type; algebras can behave locally both like a finite vector space over a finite field and like a two element boolean algebra.

1 Introduction

According to [4], there are only finitely many polynomialy nonequivalent algebras of given fixed size which generate a congruence permutable (CP) Fregean variety. Because (in CP Fregean) the clone of polynomials is determined by the congruence lattice suplemented by the commutator operator, it is easy to check that on a three element universe there are exactly four such nonequivalent algebras. Each of them can be obtained by taking an appropriate reduct of a three-element Heyting algebra. Two of those are from well known classes: equivalential algebras and Brouwerian semilattices, however the other two are not. We denote those $\mathbf{R}$ and $\mathbf{D}$. An interesting property of $\mathbf{R}, \mathbf{D}$ is that unlike equivalential algebras or Brouwerian Semilattices they both have a mixed type in the language of Tame Congruence Theory [1]. This lead to research of varieties generated by $\mathbf{R}$ and $\mathbf{D}$ done by Sławomir Przybyło in his PhD thesis (published in [5, 6]).

Because $\mathbf{R}$ can be obtained as a $(\cdot, \widehat{\wedge})$ reduct of a Heyting algebra we went on to investigate the class $\mathcal{H}$ of all $(\cdot, \widehat{\wedge})$-subreducts (of which $\mathcal{V}(\mathbf{R})$ is a proper subclass). Similar research was already done for classes of $(\cdot, \neg)$ and $(\cdot, \neg\neg)$ subreducts leading to a characterization of the first one as a quasivariety and the other as a variety [3, 2]. We follow a similar path by first "guessing" an equational class and then incrementaly showing its properties until we arrive at a conclusion that it is in fact the class of all subreducts.

2 EARS

Definition 1. An algebra $\mathbf{A} = (A, \cdot, \widehat{\wedge})$ with two binary operations is called an *equivalential algebra with regular semilattice* or *EARS* if a series of identities are satisfied for any $x, y, z \in A$. Before we write them down, for the sake of brevity, we extend our language by adding an unary operation $r(x) = x\widehat{\wedge}x$ and adopt a convention that $\cdot$ is associating to the left $(xyz = (xy)z)$. The identities are as follows:

E1. $xxy \approx y$;

E2. $xyzz \approx (xz)(yz)$;

E3. $xy(xzz)(xzz) \approx xy$;

S1. $r(x)\widehat{\wedge}y \approx x\widehat{\wedge}y$;

S2. $x\widehat{\wedge}y \approx y\widehat{\wedge}x$;

S3. $(x\widehat{\wedge}y)\widehat{\wedge}z \approx x\widehat{\wedge}(y\widehat{\wedge}z)$;

M1. $r(x)yy \approx r(x)$;

M2. $r(xx) \approx xx$;

M3. $x(y\widehat{\wedge}z)(y\widehat{\wedge}z) \approx xr(z)r(z)r(y)r(y)$;

M4. $(x\widehat{\wedge}z)(y\widehat{\wedge}z)r(z) \approx (xy)\widehat{\wedge}z$;

M5. $xr(x)r(x) \approx x$.

Identities E1-E3 make $(A, \cdot)$ an equivalential algebra and identities S1-S3 impose a semilattice structure on $(r(A), \widehat{\wedge})$. The remaining five identities describe how those two objects are mixed together.

Of course $\mathcal{H}$ is a subclass of the variety of all EARS $\mathcal{V}_{EARS}$. We will start with some basic properties of the operations and congruences of EARS and then show the following facts:

Lemma 1. $\mathcal{V}_{EARS}$ is congruence orderable.

As 1-regularity and congruence permutability is preserved by reducts it follows that

Lemma 2. $\mathcal{V}_{EARS}$ is congruence permutable Fregean.

Lemma 3. $\mathcal{V}_{EARS}$ is locally finite.

And the main result

Theorem 1. *Every finite EARS is in $\mathcal{H}$, which leads to $\mathcal{V}_{EARS} = \mathcal{H}$.*

We will also present some results about the commutator in EARS and the structure of subvarieties of $\mathcal{V}_{EARS}$.

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