The $(\infty, 2)$-category theory of internal $(\infty, 1)$-categories

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Abstract submission for TACL 2024. Internal category theory is the study of the 2-category $\mathbf{Cat}(\mathcal{C})$ of internal categories, internal functors and internal natural transformations respective a base category $\mathcal{C}$. The Yoneda embedding $y\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ can be understood as an externalization functor of finite limit structures in $\mathcal{C}$ (made precise in [5]); it can be shown to induce an embedding

$$y_*\colon \mathbf{Cat}(\mathcal{C}) \to \mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Cat})$$

of 2-categories, at times also referred to as the externalization functor of the category $\mathcal{C}$ (see e.g. [2]). On the flipside, internalization is the practice of reflecting properties and indexed structures along y_* whenever possible.

Given an ∞-categorical base $\mathcal{C}$ (usually an ∞-topos or at least left exact), the ∞-category $\mathrm{Cat}_\infty(\mathcal{C})$ of $\mathcal{C}$-internal ∞-categories is well-known, and internal constructions of internal ∞-categories are pervasive in the higher categorical literature. A systematic study of internal ∞-category theory as such however is not; indeed a definition of an according ∞-categorical enrichment $\mathbf{Cat}_\infty(\mathcal{C})$ by hand is much less tangible than in the ordinary case. Thus, following [6], in this talk we instead discuss the ∞-categorical externalization functor

$$y_*\colon \mathrm{Cat}_\infty(\mathcal{C}) \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty)$$

first, and use it to *define* the $(\infty, 2)$-category $\mathbf{Cat}_\infty(\mathcal{C})$ of $\mathcal{C}$-internal ∞-categories as embedded in the $(\infty, 2)$-category $\mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Cat}_\infty)$ of $\mathcal{C}$-indexed ∞-categories. We show various formal $(\infty, 2)$-categorical closure properties of $\mathbf{Cat}_\infty(\mathcal{C})$ under the assumption of various suitable $(\infty, 1)$-categorical closure properties of $\mathcal{C}$. The main theorem states that the $(\infty, 2)$-category $\mathbf{Cat}_\infty(\mathcal{C})$ is a full sub-∞-cosmos of $\mathbf{Fun}(\mathcal{C}^{op}, \mathbf{Cat}_\infty)$ which is closed under all limits (and exponentials) whenever $\mathcal{C}$ is complete (and cartesian closed). It thus defines a (cartesian closed) ∞-cosmos in the sense of [3]. This means that a plethora of indexed ∞-categorical constructions defined over a collection of internal ∞-categories indexed over such a base $\mathcal{C}$ can be internalized in $\mathcal{C}$ automatically. We furthermore characterize the objects of $\mathbf{Cat}_\infty(\mathcal{C})$ by means of a Yoneda lemma that expresses indexed diagrams of internal shape over $\mathcal{C}$ in terms of an ∞- categorical totalization, and discuss applications.

Lastly, we relate the general theory developed to this point to results in the model categorical literature. We show that every model category $\mathbb{M}$ gives rise to a "hands-on" ∞-cosmos $\mathbf{Cat}_\infty(\mathbb{M})$ (of not-necessarily cofibrant objects) directly by restriction of the Reedy model structure on $\mathbb{M}^{\Delta^{op}}$. We then define an according right derived model categorical externalization functor, and use it to show that the ∞-categorical and the model categorical constructions correspond to one another whenever $\mathcal{C}$ is presentable and $\mathbb{M}$ is a suitable presentation thereof. This shows that the theory presented in this talk recovers as special cases various well-known constructions in the model categorical literature. This for instance includes Dugger's simplicial replacements of model categories [1], Toën's framework for theories of $(\infty, 1)$-categories [7], as well as Riehl and Verity's ∞-cosmoses of Rezk-objects [4] among others.

References

[1] D. Dugger, *Replacing model categories with simplicial ones*, Trans. Am. Math. Soc. 353 (2001), no. 12, 5003–5027.

[2] B. Jacobs, *Categorical Logic and Type Theory*, Studies in Logic and the Foundations of Mathematics, vol. 141, Elsevier Science B.V., 1999.

[3] E. Riehl and D. Verity, *Elements of ∞-Category Theory*, Cambridge Studies in Advanced Mathematics, vol. 194, Cambridge University Press, 2022.

[4] E. Riehl and D. Verity, *Fibrations and Yoneda's lemma in an ∞-cosmos*, J. Pure Appl. Algebra 221 (2017), no. 3, 499–564.

[5] D. Scott, *Relating theories of the λ-calculus*, To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism, Academic Press, 1980.

[6] R. Stenzel, The $(\infty, 2)$-category of internal $(\infty, 1)$-categories, preprint arXiv:2402.01396, 2024.

[7] B. Toën, *Vers une axiomatisation de la théorie des catégories supérieures*, K Theory 34 (2005), no. 3, 233-263.