Bi-Intermediate Logics of Co-Trees: Local Finiteness and Decidability

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Bi-intuitionistic logic bi-IPC is the conservative extension of (propositional) intuitionistic logic IPC obtained by adding a new binary connective $\leftarrow$ to the language, called the *co-implication*, which behaves dually to $\rightarrow$. In this way, bi-IPC reaches a symmetry, which IPC lacks, between the connectives $\wedge, \top, \rightarrow$ and $\vee, \bot, \leftarrow$, respectively. Furthermore, thanks to the co-implication, bi-IPC achieves significantly greater expressivity than IPC. For instance, if the points of a Kripke frame $\mathfrak{M}$ are interpreted as states in time, the language of bi-IPC is expressive enough to talk about the past, something that is not possible in IPC. This feature is captured by the transparent interpretation of co-implication provided by the Kripke semantics of bi-IPC [11], since $\mathfrak{M}, x \models \phi \leftarrow \psi$ iff $\exists y \leq x$ ($\mathfrak{M}, y \models \phi$ and $\mathfrak{M}, y \not\models \psi$).

The greater symmetry of bi-IPC when compared to IPC is reflected in the fact that bi-IPC is algebraized in the sense of [3] by the variety bi-HA of *bi-Heyting algebras* [10], i.e., Heyting algebras whose order duals are also Heyting algebras. As a consequence, the lattice of *bi-intermediate logics* (i.e., consistent axiomatic[1] extensions of bi-IPC) is dually isomorphic to that of nontrivial varieties of bi-Heyting algebras. The latter, in turn, is not only amenable to the methods of universal algebra, but also from those of duality theory, since the category of bi-Heyting algebras is dually equivalent to that of *bi-Esakia spaces* [5], see also [1].

In [2], we began studying extensions of the *bi-intuitionistic Gödel-Dummett logic* bi-GD $:=$ bi-IPC $+(p \rightarrow q) \vee (q \rightarrow p)$, the bi-intermediate logic axiomatized by the *Gödel-Dummett axiom* (also known as the *prelinearity axiom*). Over IPC, this formula axiomatizes the well-known *intuitionistic linear calculus* LC $:=$ IPC $+ (p \rightarrow q) \vee (q \rightarrow p)$ (see, e.g., [4, 6, 8, 7]). While both logics are Kripke complete with respect to the class of *co-trees* (i.e., posets with a greatest element and whose principal upsets are chains), notably, the properties of these logics diverge significantly. For example, while LC has only countably many extensions, all of which are locally finite, we proved that bi-GD is not locally finite and has continuum many extensions. Moreover, LC is also Kripke complete with respect to the class of chains, whereas we showed that the bi-intermediate logic of chains is a proper extension of bi-GD (namely, the one obtained by adding the *dual Gödel-Dummett axiom* $\neg[(q \leftarrow p) \wedge (p \leftarrow q)]$ to bi-GD). This strongly suggest that the language of bi-IPC is more appropriate to study tree-like structures than that of IPC (since we work with a symmetric language, all of our results can be dualized to the setting of trees in a straightforward manner).

One notable extension of bi-GD is $\mathit{Log}(FC) := \{\varphi \colon \forall n \in \mathbb{Z}^+ \ (\mathfrak{C}_n \models \varphi)\}$, the *logic of the finite combs* (i.e., finite co-trees whose shape resembles that of a comb, see Figure 1). We showed in [2] that if L is an extension of bi-GD, then L is locally finite iff $L \nsubseteq \mathit{Log}(FC)$. Consequently, $\mathit{Log}(FC)$ is the only pre-locally finite extension of bi-GD (i.e., it is not locally finite, but all of its proper extensions are so). More recently, we found a finite axiomatization for $\mathit{Log}(FC)$, using Jankov and subframe formulas (the theories of these types of formulas for bi-GD were developed in [2, 9]). Since, by definition, this logic has the finite model property, we

[1] From now on we will use *extension* as a synonym of *axiomatic extension*.

can conclude that the problem of determining if a recursively axiomatizable extension of bi-GD is locally finite is decidable.

In this talk, we will cover the main steps of our recent proof. Namely, we will provide a characterization of the bi-Esakia duals of the finitely generated subdirectly irreducible algebras which validate bi-GD plus three particular Jankov formulas and one subframe formula. We will then present a combinatorial method we developed which can be used to show that the variety generated by the aforementioned algebras has the finite model property. This allows us to infer that $Log(FC)$ coincides with the extension of bi-GD axiomatized by the above mentioned Jankov and subframe formulas.

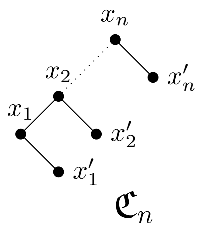

Figure 1: The n-comb $\mathfrak{C}_n$.

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