Tensor Product in the Category of Effect Algebras and Related Categories

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Tensor product of effect algebras was studied in various articles, e.g. [JM],[JP],[Gu]. However, there are few results about which constructions and functors involving effect algebras preserve tensor products.

An important property of the category of effect algebras **EA** is that the monoidal unit and the initial object coincide. Consequently, we can consider tensoring with a fixed effect algebra E as a functor:

$$E \otimes - \colon \mathbf{EA} \to E \downarrow \mathbf{EA} \tag{1}$$

which sends an effect algebra F to a homomorphism $E \to E \otimes F$, $a \mapsto 1 \otimes a$.

Theorem 1. *For an effect algebra E, the functor $E \otimes -$ from $\mathbf{EA}$ to $E \downarrow \mathbf{EA}$ which sends F to a morphism $\iota_{E,F} \colon E \to E \otimes F$ ($a \mapsto a \otimes 1$) admits a right adjoint $[E, -]_{-}$.*

Corollary 2. *Let $\mathcal{D}$ be a small connected category and $E \in \mathbf{EA}$. The functor $E \otimes - \colon \mathbf{EA} \to \mathbf{EA}$ preserves all colimits over $\mathcal{D}$.*

It turns out that several categories around **EA** share the same property. In particular, the category of ordered Abelian groups with strong unit $\mathbf{POG}_u$ and the category of partial bounded commutative monoids $\mathbf{PCM}_b$ satisfy theorems analogous to Theorem 1. Category **EA** sits between these two categories via a pair of adjunctions:

$$\mathbf{PCM}_b \underset{L}{\overset{i}{\rightleftarrows}} \mathbf{EA} \underset{\mathrm{Gr}}{\overset{\Gamma}{\rightleftarrows}} \mathbf{POG}_u \qquad (L \dashv i,\ \mathrm{Gr} \dashv \Gamma) \tag{2}$$

Theorem 3. *For any $X, Y \in \mathbf{PCM}_b$ and $E, F \in \mathbf{EA}$ we have*

$$L(X \otimes Y) \cong L(X) \otimes L(Y) \text{ and } \mathrm{Gr}(E \otimes F) \cong \mathrm{Gr}(E) \otimes \mathrm{Gr}(F). \tag{3}$$

Where functors L and Gr are from (2) and the tensor products are computed in the appropriated categories.

In the case of Gr, we have even stronger result:

Theorem 4. *The left adjoint Gr in (2) extend to a strong monoidal functor.*

In the case of $\mathrm{Gr} \colon \mathbf{EA} \to \mathbf{POG}_u$, the isomorphism (3) follows from (up to isomorphism) commutativity of the diagram (4), where E is any effect algebra and $A = \mathrm{Gr}(E)$.

$$\begin{array}{ccc} E \downarrow \mathbf{EA} & \xleftarrow{E \otimes -} & \mathbf{EA} \\ \Big\downarrow{\scriptstyle \mathrm{Gr}} & & \Big\downarrow{\scriptstyle \mathrm{Gr}} \\ A \downarrow \mathbf{POG}_u & \xleftarrow{A \otimes -} & \mathbf{POG}_u \end{array} \tag{4}$$

The functors involved in (4) correspond to some free constructions and so are rather complicated. In the proof of commutativity, we use a trick. We move to the corresponding right adjoints (which all exist). The right adjoints all have a description in concrete terms, hence are easier to work with.

By a result in [We], the tensor product in $\mathbf{POG}_u$ does not preserve *Riesz Decomposition Property* (RDP) in general. Whereas in $\mathbf{PCM}_b$, the tensor product does preserves (RDP). The case of effect algebras was an open problem for a while. Thanks to Theorem 3, we can lift the contra-example, which works in $\mathbf{POG}_u$, to $\mathbf{EA}$.

Theorem 5. *In* $\mathbf{EA}$*, tensor product does not preserves Riesz Decomposition Property in general.*

Theorem 5 has the following implications:

- Computing tensor products in $\mathbf{EA}$ is rather hard, in the sense we cannot control it using (RDP). That is in contrast to the construction of a universal group (functor Gr), which preserves (RDP).
- The functor $L\colon \mathbf{PCM}_b \to \mathbf{EA}$, which essentially forces cancellation property, does not preserve (RDP).

It is not well understood which tensor products are preserved by the right adjoints in (2). However, it is proved in [Pu] that functor Γ preserves the tensor product of $(\mathbb{R}, 1)$ with itself, that is

$$\Gamma(\mathbb{R} \otimes \mathbb{R}, 1 \otimes 1) \cong [0,1] \otimes [0,1]. \tag{5}$$

The question of whether the embedding $i\colon \mathbf{EA} \hookrightarrow \mathbf{PCM}_b$ preserves the tensor product of the real unit interval $[0,1]$ (seen as an effect algebra) with itself leads to an interesting combinatorial problem. In the case of $\mathbf{PCM}_b$, it holds that two tensors $a_1 \otimes b_1 + \cdots + a_n \otimes b_n$ and $c_1 \otimes d_1 + \cdots + c_m \otimes d_m$ in $[0,1] \otimes [0,1]$ are equal if and only if we can represent the two tensors as two orthogonal polygons $\mathcal{P}_1$ and $\mathcal{P}_2$ inside the unit square $[0,1] \times [0,1]$, and there is an orthogonal dissection between $\mathcal{P}_1$ and $\mathcal{P}_2$. By a result in [Ep], there is a full Dehn invariant for this kind of dissection. We have used this result to show that computing the tensor product of the real unit interval with itself as a partial monoid in $\mathbf{POG}_b$ and as an effect algebra in $\mathbf{EA}$ is essentially equivalent.

References

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