On Three-valued Coalgebraic Cover Modalities*

Chun-Yu Lin and Marta Bílková

Institute of Computer Science, the Czech Academy of Sciences ,Prague
chunyumaxlin@gmail.com & bilkova@cs.cas.cz

Coalgebraic logic proposed by Moss uses a single modality ∇_T with a set endo-functor T as its arity and T-coalgebras as structural frames [11]. Finitary version of Moss' coalgebraic logic has found applications in logic and automata theory [9], its soundness and completeness has been established in a form of Hilbert-style axiomatization [8], and the Gentzen-syle sequent calculi [4]. An adaptation of Moss' coalgebraic logic to a many-valued context, where formulas are evaluated in a given algebra of truth values, has been explored *e.g.* in [2] and [3]. Nevertheless, proof theory for many-valued coalgebraic cover modality has not been discussed yet. Our aim is to bridge this gap by proposing a Gentzen-style sequent calculus for a three-valued coalgebraic cover modality, expanding Kleene logic. Besides, we will also touch upon utilizing the abstract approach as in [10] and propose possible axioms for Hilbert-style systems over semi-primal algebras such as 3-valued Łukasiewicz chain.

We start with choosing, as the propositional base, Strong Kleene logic (K_3), Weak Kleene logic (WK_3), which arise from different algebras (matrices) on the three values $\{1, n, 0\}$, with varying interpretation of the third value n (undefined, nonsensical, paradoxical) [7, 6]. Consequence relations of these logics can be closely related to classical consequence. In case of WK_3 where the three-element algebra is not a lattice and the third value n is infectious it is done using certain variable containment conditions. This allows for a natural adaptation of classical sequent calculus where some rules use variable containment side conditions [5]. We show how these conditions can be modalized and use it to built on sequent calculi for coalgebraic cover modality.

As a starting example, consider $\mathcal{P}$ to be the (covariant) power set functor and $\mathcal{P}_\omega$ be the finitary power set functor. Let $\mathcal{L}_{K_3}$ be the following language:

$$\varphi \coloneqq p \mid \bigvee \Phi \mid \bigwedge \Phi \mid \neg\varphi \mid \nabla\alpha \mid \Delta\alpha$$

where $p \in Prop$, a set of propositional variables, and $\Phi, \alpha \in P_\omega \mathcal{L}_{K_3}$. The set $Var_i(\Phi)$ denotes propositional variables within formulas of modal depth i in Φ, and $Base^{\mathcal{P}_\omega}_{\mathcal{L}_{K_3}}(\alpha)$ is defined as $\bigcap\{X \subseteq_\omega \mathcal{L}_{K_3} \mid \alpha \in \mathcal{P}_\omega X\}$. The semantics for the logical connectives of $\mathcal{L}_{WK_3}$ can be defined using the truth tables in Weak Kleene logic. The semantics for $\nabla\alpha$ is defined as follows:

Definition 1. *Let S be a set. For a coalgebra $\sigma : S \to \mathcal{P}(S)$ together with the atomic evaluation $ev : S \times Prop \to \{0, 1, n\}$*

$$s \Vdash^w_\sigma \nabla\alpha \coloneqq \sigma(s)\hat{\mathcal{P}}(\Vdash^w_\sigma)(\alpha) = \bigwedge_{t\in\sigma(s)} \bigvee_{a\in\alpha} t \Vdash^w_\sigma a \wedge \bigwedge_{a\in\alpha} \bigvee_{t\in\sigma(s)} t \Vdash^w_\sigma a$$

where $\hat{\mathcal{P}}(\Vdash^w_\sigma)$ is the power set relation lifting of $\Vdash^w_\sigma$.

The infectious property of Weak Kleene logic implies that if there exist some $t \in \sigma(s)$ and $a \in \alpha$ such that $t \Vdash^w_\sigma a = n$ then $s \Vdash^w_\sigma \nabla\alpha = n$. Otherwise the $\Vdash^w_\sigma$ relation acts the same as in the classical case. The Genzen sequent calculi GWK_3 employs the following modal depth-specific side conditions as in [5]. For example, the (¬-r) rule would now become:

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$$\frac{\Gamma, a \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma, \neg a} \,(\neg\text{-r}),\ \forall i \leq m,\ Var_i(a) \subseteq Var_i(\Gamma)$$

$$\frac{\{A_L^\Phi \Rightarrow A_R^\Phi \mid \Phi \in SRD(\Gamma \uplus \Sigma)\}}{\{\nabla\alpha \mid \alpha \in \Gamma\} \Rightarrow \{\Delta\beta \mid \beta \in \Sigma\}} \,\forall \Phi . A^\Phi \in Base(\Phi), \forall i \leq n,\ Var_i(A_L^\Phi) \subseteq Var_i(A_R^\Phi)$$

where m is the maximum modal depth of formulas in Γ, and n is a maximum modal depth of formulas in $\Gamma \cup \Sigma$. In this talk, we will discuss how to obtain a Gentzen system for the coalgebraic cover modality over Weak Kleene logic.

For the Strong Kleene logic, the semantics for logical connectives in $\mathcal{L}_{K_3}$ is defined via the truth tables in Strong Kleene logic, and the modal formula $\nabla\alpha$ is defined similarly to Definition 1. The Genzen sequent calculus GK_3 is based on GWK_3, obtained by removing all the side conditions and adding six negation related rules [1]. We will show how to extend the calculus with the ∇-modality rules. We will then discuss soundness and completeness of the resulting calculi. As [10] indicates, completeness can be lifted from the classical logic to the many-valued logic in case the algebras are semi-primal. Nevertheless, since the semantic for Weak Kleene logic is not semi-primal, the approach in [10] is not feasible here.

In the end of this talk, we will briefly address the problem of axiomatizing semi-primal algebra-valued coalgebraic logic by demonstrating when modifying the modal axioms (∇1)-(∇4) in [8] and adding the following axioms results in a sound Hilbert-style axiomatic system:

$$\tau_v(\nabla\Phi) \equiv \nabla T(\tau_v)(\Phi),$$

where v are elements of semi-primal algebras $\mathbb{A}$ and for τ_v are unary operations defined by

$$\tau_v(x) = \begin{cases} 1, & \text{if } x \geq v \\ 0, & \text{if } x \ngeq v. \end{cases}$$

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