Degrees of incompleteness of implicative logics: the trichotomy theorem

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The degree of incompleteness of a logic L is the cardinality of the set of logics with the same Kripke frames as L [3]. Blok's celebrated *dichotomy theorem* states that every normal modal logic has degree of incompleteness either 1 or $2^{\aleph_0}$ [1]. The aim of this talk is to prove a *trichotomy theorem* for the degrees of incompleteness of the *implicative logics*, i.e., the axiomatic extensions of the implicative fragment $\mathsf{IPC}_{\to}$ of the intuitionistic logic IPC. In particular, we will prove that the degree of incompleteness of any implicative logic is 1, $\aleph_0$, or $2^{\aleph_0}$. Notably, the degree of incompleteness of an implicative logic coincides with its *degree of the finite model property*, as defined in [4].

In what follows, we make the above statement precise. A formula of the *intuitionistic propositional calculus* IPC is said to be *implicative* when it contains no connective other than $\to$.

Definition 1. The *implicative fragment* of IPC is the set

$$\mathsf{IPC}_{\to} \coloneqq \{\varphi \in \mathsf{IPC} : \varphi \text{ is an implicative formula}\}.$$

Notably, $\mathsf{IPC}_{\to}$ coincides with the set of implicative formulas φ such that $\mathsf{Hil} \vDash \varphi$, where Hil is the variety of *Hilbert algebras*, i.e., the class of subalgebras of the *implicative reducts* $\langle A; \to \rangle$ of Heyting algebras [2]. Since $x \to x$ is a constant term in every Hilbert algebra, we will use the shorthand $1 \coloneqq x \to x$.

Definition 2. An *implicative logic* is a set of implicative formulas containing $\mathsf{IPC}_{\to}$ that, moreover, is closed under modus ponens and uniform substitutions.

When ordered under the inclusion relation, the set of implicative logics forms a complete lattice $\mathrm{Ext}(\mathsf{IPC}_{\to})$ which is dually isomorphic to the lattice $\Lambda(\mathsf{Hil})$ of varieties of Hilbert algebras. This dual isomorphism is witnessed by the maps $\mathsf{Var}(-)$ and $\mathsf{Log}(-)$ defined for every $\mathsf{L} \in \mathrm{Ext}(\mathsf{IPC}_{\to})$ and $\mathsf{V} \in \Lambda(\mathsf{Hil})$ as

$$\begin{aligned} \mathsf{Var}(\mathsf{L}) &\coloneqq \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \vDash \mathsf{L}\}; \\ \mathsf{Log}(\mathsf{V}) &\coloneqq \{\varphi : \varphi \text{ is an implicative formula such that } \mathsf{V} \vDash \varphi\}. \end{aligned}$$

Given a poset $\boldsymbol{X}$ and a set Γ of implicative formulas we write $X \Vdash \Gamma$ when Γ is valid in $\boldsymbol{X}$, viewed as an intuitionistic Kripke frame.

Definition 3. The *span* of an implicative logic L is the set

$$\mathsf{span}(\mathsf{L}) \coloneqq \{\mathsf{L}' \in \mathrm{Ext}(\mathsf{IPC}_{\to}) : \boldsymbol{X} \Vdash \mathsf{L} \text{ iff } \boldsymbol{X} \Vdash \mathsf{L}', \text{ for every poset } \boldsymbol{X}\}.$$

Furthermore, the *degree of incompleteness* of L is $\mathsf{deg}(\mathsf{L}) \coloneqq |\mathsf{span}(\mathsf{L})|$.

Before stating our main result characterising the degree of incompleteness of the implicative logics, we need to introduce two classes of varieties. To this end, recall that a subset F of a Hilbert algebra $\boldsymbol{A}$ is an *implicative filter* if it contains 1 and for every $\{a, b\} \subseteq A$, if $\{a, a \to b\} \subseteq F$, then $b \in F$.

Definition 4. Given $n \in \mathbb{N}$, we say that a Hilbert algebra $\boldsymbol{A}$ has *depth* $\leqslant n$ when the poset of its meet irreducible implicative filters does not contain $(n+1)$-elements chains. Then, the following is a variety:

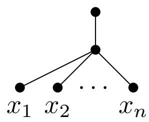
$$\mathsf{D}_n \coloneqq \{\boldsymbol{A} \in \mathsf{Hil} : \boldsymbol{A} \text{ has depth} \leqslant n\}.$$

In order to define the second class of varieties, with every poset $\boldsymbol{X} = \langle X; \leqslant \rangle$ with maximum $\top$ we associate a binary operation $\to$ on X defined by the rule

$$x \to y \coloneqq \begin{cases} \top & \text{if } x \leqslant y; \\ y & \text{otherwise.} \end{cases}$$

Then, $\mathsf{H}(\boldsymbol{X}) \coloneqq \langle X; \to \rangle$ is a Hilbert algebra with underlying partial order $\leqslant$. Lastly, we denote the smallest variety containing a class of algebras K by $\mathbb{V}(\mathsf{K})$.

Definition 5. For each $n \in \mathbb{Z}^+$ let $\boldsymbol{B}_n \coloneqq \mathsf{H}(\mathbb{B}_n)$, where $\mathbb{B}_n$ is the poset depicted below:



Furthermore, let

$$\mathsf{B}_n \coloneqq \mathbb{V}(\boldsymbol{B}_n) \quad \text{and} \quad \mathsf{B}_\omega \coloneqq \mathbb{V}(\{\boldsymbol{B}_n : n \in \mathbb{Z}^+\}).$$

Our main result takes the following form:

Trichotomy Theorem. *The following conditions hold for an implicative logic* L*:*

(i) $\mathsf{deg}(\mathsf{L}) = 1$ *if and only if* $\mathsf{L} = \mathsf{IPC}_{\to}$ *or* $\mathsf{L} = \mathsf{Log}(\mathsf{D}_n)$ *for some* $n \in \mathbb{N}$*;*

(ii) $\mathsf{deg}(\mathsf{L}) = \aleph_0$ *if and only if* $\mathsf{L} = \mathsf{Log}(\mathsf{B}_\omega)$ *or* $\mathsf{L} = \mathsf{Log}(\mathsf{B}_n)$ *for some* $n \in \mathbb{Z}^+$*;*

(iii) $\mathsf{deg}(\mathsf{L}) = 2^{\aleph_0}$ *otherwise.*

We remark that the problem of determining which are the degrees of incompleteness of intermediate logics is an outstanding open problem and hope that this talk will stimulate research in this direction.

References

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