Meet-irreducible elements in the poset of all logics

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In [2, 4, 3] the so-called *poset of all logics* is introduced and compared with the lattice Var of interpretability types of varieties (see, e.g., [1]). Roughly speaking, a variety V is *interpretable into* a variety W when W is term-equivalent to some variety, whose reducts (in a smaller language) belong to V. The interpretability relation for logics can be defined analogously, although it requires some tools from abstract algebraic logic (see, e.g., [2]).

More precisely, a (logical) matrix $(\mathscr{A}, F)$ is said to be a *model* of a logic $\vdash$ when F is a deductive filter of $\vdash$ on $\mathscr{A}$. In addition, if the unique congruence of $\mathscr{A}$ that does not glue an element of F to one of $A - F$ is the identity, we say that $(\mathscr{A}, F)$ is *reduced*. A matrix is a *Suszko model* of a logic $\vdash$ if it is isomorphic to a subdirect product of reduced models of $\vdash$. We denote by $\mathrm{Mod}^{\equiv}(\vdash)$ the class of all the Suszko models of $\vdash$ [2].

Let $\vdash$ be a logic. The set of connectives of $\vdash$ will be denoted by $\mathscr{L}(\vdash)$ and the set of terms of $\vdash$ with countably many variables by $\mathscr{T}(\vdash)$. Given two logics $\vdash$ and $\vdash'$, we say that a map $\tau \colon \mathscr{L}(\vdash) \to \mathscr{T}(\vdash')$ is a *translation* when it sends n-ary connectives to n-ary terms. In this case, with every algebra $\mathscr{A}$ in the language of $\vdash'$ we can associate an algebra $\mathscr{A}^{\tau}$ in the language of $\vdash$ defined as follows:

$$\mathscr{A}^{\tau} := (A, \{\tau(f)^{\mathscr{A}} : f \in \mathscr{L}(\vdash)\}).$$

We say that $\vdash$ is *interpretable* into $\vdash'$, in symbols $\vdash \leqslant \vdash'$, when there exists a translation $\tau \colon \mathscr{L}(\vdash) \to \mathscr{T}(\vdash')$ such that

$$(\mathscr{A}, F) \in \mathrm{Mod}^{\equiv}(\vdash') \text{ implies that } (\mathscr{A}^{\tau}, F) \in \mathrm{Mod}^{\equiv}(\vdash).$$

Two logics $\vdash$ and $\vdash'$ are said to be *equi-interpretable* when $\vdash \leqslant \vdash' \leqslant \vdash$. We denote the equivalence class of all the logics that are equi-interpretable with $\vdash$ by $[\![\vdash]\!]$. Note that $\leqslant$ is a preorder on the class of all logics. The *poset of all logics* Log is the corresponding poset, whose elements are precisely the classes $[\![\vdash]\!]$[1]. Given two logics $\vdash$ and $\vdash'$, we write $[\![\vdash]\!] \leqslant [\![\vdash']\!]$ iff $\vdash \leqslant \vdash'$.

In [2] it is shown that even if Log has infima of families indexed by arbitrarily large sets, it may lack binary suprema (this is possible because its universe is not a set). Infima in Log can be described as follows. The *non-indexed product* of a family of algebraic languages $\{\mathscr{L}_i \mid i \in I\}$ is the algebraic language $\bigotimes_{i \in I} \mathscr{L}_i$ whose n-ary symbols are of the form $(\varphi_i(\bar{x}))_{i \in I}$, where each $\varphi_i(\bar{x})$ is an n-ary term of $\mathscr{L}_i$. Moreover, the *non-indexed product* of a family $\{\mathscr{A}_i \mid i \in I\}$, where each $\mathscr{A}_i$ is a $\mathscr{L}_i$-algebra, is the $\bigotimes_{i \in I} \mathscr{L}_i$-algebra $\bigotimes_{i \in I} \mathscr{A}_i$, whose universe is $\prod_{i \in I} A_i$ and whose n-ary symbols $(\varphi_i(x_1, \dots, x_n))_{i \in I}$ are interpreted as follows:

$$(\varphi_i(x_1, \dots, x_n))_{i \in I}^{\bigotimes_{i \in I} \mathscr{A}_i}(\bar{a}_1, \dots, \bar{a}_n) := (\varphi_i^{\mathscr{A}_i}(\bar{a}_1(i), \dots, \bar{a}_n(i)))_{i \in I}.$$

Similarly, the *non-indexed product* of a family of matrices $\{(\mathscr{A}_i, F_i) \mid i \in I\}$ is the matrix $(\bigotimes_{i \in I} \mathscr{A}_i, \prod_{i \in I} F_i)$. Lastly, the *non-indexed product* of a family $\{\vdash_i \mid i \in I\}$ of logics is the logic $\bigotimes_{i \in I} \vdash_i$ in the language $\bigotimes_{i \in I} \mathscr{L}_i$ induced by the class of matrices $\bigotimes_{i \in I} \mathrm{Mod}^{\equiv}(\vdash_i)$. It turns out that $[\![\bigotimes_{i \in I} \vdash_i]\!]$ is the infimum of

[1] Although strictly speaking the universe of Log is not a set (and, therefore, Log is not a poset in the traditional sense), our results on this structure can be effortlessly rephrased in ZFC (see, e.g., [2]).

$\{[\![\vdash_i]\!] : i \in I\}$ in Log [2, Thm. 4.6].

The aforementioned description of infima allows us to introduce a notion of meet-irreducibility for arbitrary logics. More precisely, we say that a logic $\vdash$ is *meet-irreducible* when $[\![\vdash]\!]$ is a meet-irreducible element of Log, i.e., for every pair of logics $\vdash_1$ and $\vdash_2$,

$$[\![\vdash_1 \otimes \vdash_2]\!] = [\![\vdash]\!] \text{ implies that either } \vdash_1 \leqslant \vdash \text{ or } \vdash_2 \leqslant \vdash .$$

Our main result provides a sufficient condition for the meet-irreducibility of a given logic. We say that a model of a logic $\vdash$ is *trivial* when it is either of the form $(\mathbf{1}, \{1\})$ or $(\mathbf{1}, \varnothing)$, where $\mathbf{1}$ is the trivial $\mathscr{L}(\vdash)$-algebra. On the other hand, recall that a class of similar matrices $\mathbb{K}$ has the *joint embedding property* (JEP) if for every set X of nontrivial members of $\mathbb{K}$ there exists some $(\mathscr{A}, F) \in \mathbb{K}$ in which every member of X embeds.

Theorem 1. *Every logic with theorems $\vdash$ satisfying the following conditions is meet-irreducible:*

(1) $\mathrm{Mod}^{\equiv}(\vdash)$ *has the JEP;*

(2) The nontrivial members of $\mathrm{Mod}^{\equiv}(\vdash)$ *have substructures of prime cardinality;*

(3) The nontrivial members of $\mathrm{Mod}^{\equiv}(\vdash)$ *lack trivial substructures.*

As a consequence, every intermediate logic is meet-irreducible and so are some prominent modal logics such as the global consequence of the normal modal logic S4.

It is natural to compare the above result with a well-known sufficient condition for meet-primeness in the lattice of interpretability types of varieties which states that, if V is the variety generated by a nested countable union of varieties V_n, where each V_n is generated by a finite algebra of prime cardinality, then V is meet-prime in Var [1, Prop. 18]. During the talk we will also discuss a variant of this observation in the context of logics (as opposed to varieties).

References

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