Barr exactness in classes of locally finite, transitive and reflexive Kripke frames

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Kripke frames (sets equipped with a binary relation) are one of the most popular semantics of modal logics (see [4] for a complete overview). They form the category **KFr**, where the arrows are the so called *p-morphisms*. Images via p-morphisms are called *p-morphic images* and such images are *generated subframes* of their codomains. A Kripke frame $\mathcal{F}$ is called *locally finite* if, for each $p \in \mathcal{F}$, the smallest generated subframe containing p is finite (in literature, *image finite* Kripke frames are better known; locally finite Kripke frames are those Kripke frames whose transitive closure is image finite). We are interested in $\mathbf{KFr}_{lf}$, the full subcategory of *locally finite* Kripke frames: this subcategory is closed under coproducts (disjoint unions), generated subframes and p-morphic images. More generally, we are interested in any full subcategory $\mathcal{C} \subseteq \mathbf{KFr}_{lf}$ closed under the same operations (all colimits in $\mathcal{C}$ can be built from such operations). In [2], it has been shown that $\mathcal{C}$ is always *comonadic* over **Set**.

The algebraic semantics of modal logic is given by *modal algebras*. In the so called Thomason duality [3], $\mathbf{KFr}_{lf}$ corresponds to $\mathrm{Pro}\mathbf{MA}_f$, the category of profinite modal algebras, with suitable morphisms, which is monadic over **Set** [2] (while image finite Kripke frames are dual to the topological modal algebras whose underlying topology is a Stone topology). Topological algebras and profiniteness are strictly related to classical problems such as canonical extensions of lattice-based algebras (among them are modal algebras). More generally, for any variety $\mathbf{V}$ of modal algebras generated by its finite members $\mathbf{V}_f$, the pro-completion [6] $\mathrm{Pro}\mathbf{V}_f$ is monadic over **Set**. In the above duality, $\mathrm{Pro}\mathbf{V}_f$ corresponds to the class of locally finite Kripke frames validating the equations defining $\mathbf{V}$; the latter class has the aforementioned closure properties.

Our aim is to study categorical properties of classes of locally finite Kripke frames dual to $\mathrm{Pro}\mathbf{V}_f$, for some $\mathbf{V}$. In particular, we want to characterize regularity and Barr exactness, at least under the assumption that the Kripke frames are transitive. Indeed, it is possible to prove that: (i) such classes have all limits (being the ind-completion of the class of finite Kripke frames belonging to it [2]) and (ii) under the assumption of transitivity, the usual image factorization gives an (extremal epi, mono)-factorization. Therefore, to establish regularity, it only remains to check that extremal epimorphisms are stable under pullbacks. We present a partial solution for the reflexive and transitive case.

From now on, we fix a full subcategory $\mathcal{C}$ of *reflexive and transitive* locally finite Kripke frames closed under disjoint unions, generated subframes and p-morphic images. In this case, the stability of extremal epimorphisms under pullbacks can be rephrased in terms of the dual of the *amalgamation property*. A *co-amalgamation* for a finite family $f_1, \dots, f_n$ of epimorphisms with common codomain is a family $g_1, \dots, g_n$ of epimorphisms with common domain, such that all the compositions $f_i g_i$ exist and coincide. The category $\mathcal{C}$ is said to satisfy the *co-amalgamation property* if each finite family of epimorphisms with common codomain has a co-amalgamation.

Co-amalgamation can be used to find out necessary conditions for regularity (following the classification in [5, Section 6.3], see also [8, 7]): if $\mathcal{C}$ is regular, then it is forced to contain Kripke frames that can be built using co-amalgamation and p-morphic images.

The construction of a binary product in $\mathcal{C}$ can be performed by induction following the universal model construction, well known in the modal logic literature — see [1]. This implies that the product of a pair of objects in $\mathcal{C}'$ is a generated subframe of the product computed in any $\mathcal{C}$ containing $\mathcal{C}'$. The two products might coincide, for example, when $\mathcal{C}' = \mathcal{C} \cap \mathbf{Pos}_{lf}$, where $\mathbf{Pos}_{lf}$ is the class of locally finite posets. If this is the case, $\mathcal{C}'$ is closed under pullbacks in $\mathcal{C}$, being always closed under equalizers. This observation allows us to conclude that, if $\mathcal{C}$ is regular, then all its subclasses closed under finite products in $\mathcal{C}$ must be regular; in particular, $\mathcal{C} \cap \mathbf{Pos}_{lf}$ has to be regular, too. A case analysis, based on the co-amalgamation property, shows that exctly 8 subclasses of $\mathbf{Pos}_{lf}$ are regular. Therefore, the regular $\mathcal{C}$ must intersect $\mathbf{Pos}_{lf}$ in one of the 8 classes above; applying again the co-amalgamation property, we obtain 49 possible cases.

Barr exactness can also be studied. Similarly to what happens for regularity, given two regular $\mathcal{C}' \subseteq \mathcal{C}$, with $\mathcal{C}'$ closed under finite products in $\mathcal{C}$, if $\mathcal{C}$ is exact then $\mathcal{C}'$ is exact, too. In particular, $\mathcal{C} \cap \mathbf{Pos}_{lf}$ is exact if $\mathcal{C}$ is so. After having excluded a certain number of cases, we show that $\mathcal{C}$ is exact if it only contains the empty frame, or it is one of the following:

1. $\{\mathcal{F} \mid \mathrm{ht}(\mathcal{F}) \leq 1 \ \&\ \delta^e(\mathcal{F}) \leq 1\} \cong \mathbf{Set}$;
2. $\{\mathcal{F} \mid \mathrm{ht}(\mathcal{F}) \leq 1 \ \&\ \delta^e(\mathcal{F}) \leq 2\} \cong \mathbb{Z}_2^{+}\text{-}\mathbf{Set}$;
3. $\{\mathcal{F} \mid \mathrm{ht}(\mathcal{F}) \leq 2 \ \&\ \mathrm{wt}(\mathcal{F}) \leq 1 \ \&\ \delta^i(\mathcal{F}) \leq 1 \ \&\ \delta^e(\mathcal{F}) \leq 1\} \cong \mathbb{Z}_2^{\times}\text{-}\mathbf{Set}$;

Where ht and wt give bound for cardinality of chains, resp. antichains, and δ^e and δ^i give bound for cardinality of external, resp. internal clusters.

We are currently working on a full characterization of exactness in the reflexive and transitive case and on a generalization of this characterization without the reflexivity condition. In the latter context, exactness could be encountered in some non trivial cases. An example is given by the class $\mathbf{GL}\text{-Lin}_{lf}$ of locally finite, transitive and irreflexive Kripke frames for which the restriction of the binary relation to each rooted generated subframe is a (irreflexive) linear order: $\mathbf{GL}\text{-Lin}_{lf}$ is indeed equivalent to the category of presheaves $\mathbf{Set}^{(\mathbb{N},\leq)^{\mathrm{op}}}$.

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