Localic uniform completions via Cauchy sequences

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Uniform spaces provide a general setting in which to discuss uniform continuity and completeness. A uniform space is given by a set X equipped with a filter $\mathcal{E}$ of binary relations on X, called *entourages*, satisfying certain axioms. (For instance, see [4, Chapter 9].) Entourages intuitively act like approximate equality relations. Every metric space (X, d) gives rise to a uniform space with basic entourages $E_\varepsilon = \{(x, y) \in X \times X \mid d(x, y) < \varepsilon\}$.

Recall that the completion of a metric space can be constructed as a quotient of the set of Cauchy sequences. However, this does not work for general uniform spaces. The problem is that sequences are not always 'long enough' and so we need to use *Cauchy filters* or *Cauchy nets* instead.

The pointfree approach to uniformity replaces uniform spaces with *uniform locales*. Analogously to before, a uniform locale is a locale X equipped with a filter of open sublocales of X^2 satisfying certain conditions and there is a well-developed theory of completions of uniform locales via Cauchy filters. See [2] or [1] for details.

In this talk, we will show that, in contrast to the situation with uniform spaces, the correct completion of uniform locales *can* also be obtained using Cauchy sequences. Our construction is based on the construction of the so-called 'localic completion' of metric spaces in terms of Cauchy sequences described by Vickers in [3], but generalises it to start with locales rather than sets and to use uniform rather than metric structures.

We must first construct a locale of Cauchy sequences. Usually, we would obtain this as the classifying locale of a *geometric theory* of Cauchy sequences. Recall that a Cauchy sequence in a uniform space $(X, \mathcal{E})$ is a map $s \colon \mathbb{N} \to X$ such that $\forall E \in \mathcal{E}.\ \exists N \in \mathbb{N}.\ \forall n, n' \geq N.\ (s(n), s(n')) \in E$. The problem is that this definition involves universal quantification over $\mathbb{N}$ and so is too logically complex to be described by a geometric theory. Vickers circumvents this by asking the Cauchy sequences to converge rapidly, but rapid convergence cannot be defined outside of the metric setting.

Instead we 'Skolemise' the definition to reverse the quantifiers and give $\exists m \colon \mathcal{E} \nrightarrow \mathbb{N}.\ \forall E \in \mathcal{E}.\ \forall n, n' \geq m(E).\ (s(n), s(n')) \in E$ and then include the *modulus of Cauchyness* m in the data of a Cauchy sequence. (This additional data will be discarded by the quotient step in any case.) This allows us to define a locale of *modulated Cauchy sequences*.

Finally, we construct a map from this locale to the usual completion and prove it is a well-behaved quotient map. Thus, the completion can indeed be obtained as a quotient of the locale of (modulated) Cauchy sequences.

A natural question is now: what goes wrong in the spatial setting? The problem is that, unless the uniformity has a countable base, the locale of modulated Cauchy sequences is unlikely to be spatial. The spatial construction can be understood as taking the points of this locale *before* taking the quotient. To obtain the correct completion of a uniform space we must instead take points *after* taking the quotient. Thus, the root of the pathologies that occur in the spatial setting is that taking the spectrum of a locale does not preserve quotients!

References

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