Logics for Probabilistic Dynamical Systems

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Our main objective is to investigate (discrete-time) Markov stochastic processes augmented by a dynamic mapping from the modal logic point of view. These mathematical structures, which we call *dynamic Markov processes*, are of the form $\langle \Omega, \mathcal{A}, T, f\rangle$ where $\langle \Omega, \mathcal{A}\rangle$ is a measure space, $T : \Omega \times \mathcal{A} \to [0, 1]$ is a Markov kernel and $f : \Omega \to \Omega$ is a measurable function. In this case, the triple $\langle \Omega, \mathcal{A}, T\rangle$ is called a *Markov process* on the state space $\langle \Omega, \mathcal{A}\rangle$.

In a somewhat broader context, the notion of *probabilistic (random) dynamical systems* [1] covers one of the most important classes of dynamical systems with probabilistic features. Typically, these systems contain stochastic processes, e.g. Markov processes possibly augmented by some additional dynamic structures that describe the dynamic behavior of the system. In a sense, our investigations lay in logical descriptions of certain special cases of probabilistic dynamical systems. These structures have diverse applications, from stochastic differential equations to finance and economics [4].

There are various logical approaches to modeling probability structures, among which we consider propositional modal logic. In this approach, bounds on probability are treated as modal operators. So there are countably many probability modal operators L_r, for each $r \in \mathcal{Q} \cap [0, 1]$. For a formula φ, the formula $L_r\varphi$ is interpreted as 'the probability of φ is at least r'. The resulting *modal probability logic* is denoted by PL. It is shown that this logic is decidable [7, 13]. There are numerous papers in this area dealing with axiomatization which demonstrate several completeness for PL [2, 7, 8, 10, 13] and prove some nice semantical properties [6, 12]. There is also infinitary version of PL denoted by PL_{ω_1} [3, 9, 11]. The language PL_{ω_1} extends the language PL by adding (infinite) countable conjunctions and disjunctions.

This presentation, which is based on our recent work in [5], is divided into two parts. The first part of our research is devoted to introducing the finitary *dynamic probability logic* (DPL). The language of DPL is obtained by adding a temporal-like modal operator $\bigcirc$ (denoted as dynamic operator) which describes the dynamic part of the system. We subsequently propose a Hilbert-style axiomatization for this logic and demonstrate its strong completeness for the class of all dynamic Markov processes based on standard Borel spaces[1]. To this end, we use a canonical model construction based on special maximal finitely consistent subsets of formula called *saturated sets*. This approach is inspired by the proof of strong completeness for Markovian logics in [10]. We further examine the logics of some important subclasses of dynamic Markov processes, including the class of all dynamic Markov processes of the form $\langle \Omega, \mathcal{A}, T, f\rangle$ that are *measure-preserving*, i.e., $T(w, f^{-1}(A)) = T(f(w), A)$ for each $w \in \Omega$ and $a \in \mathcal{A}$. We also present a logic for the class of all *abstract dynamical systems*, i.e. structures of the form $\langle \Omega, \mathcal{A}, \mu, f\rangle$ where $\langle \Omega, \mathcal{A}, \mu\rangle$ is a probability space and $f : \Omega \to \Omega$ is a measure-preserving function.

Our ideas naturally extend to introducing the *infinitary dynamic probability logic*. This logic, which is denoted by DPL_{ω_1}, allows countable conjunctions and disjunctions. The expressive power of DPL_{ω_1} is compatible with σ-additivity of probability measures. So within this logic,

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[1] A measure space $\langle \Omega, \mathcal{A}\rangle$ is called a *standard Borel space* if $\mathcal{A}$ is the Borel σ-algebra generated by a Polish topology on Ω.

many properties of probability can be naturally axiomatized, and hence, it is not hard to extend ideas from [3, 9] to show that there exists a weakly complete Hilbert-style axiomatization for this logic. Meanwhile, we show that whenever the logic is restricted to its countable fragments, the proposed axiomatization is strongly complete for the class of all dynamic Markov processes. We should point out that while the canonical model introduced for the proof of strong completeness for each countable fragment A of PL_{ω_1} in [9, Subsection 5.2] depends on A, we show that the canonical model of DPL can be served uniformly as a canonical model for each countable fragment of DPL_{ω_1}.

The second contribution of the present research is allocated to investigating (frame) definability of natural properties of dynamic Markov processes. We show that some dynamic properties such as measure-preserving, ergodicity, and mixing are definable within DPL and DPL_{ω_1}. Moreover, we consider the *infinitary probability logic with initial distribution* (InPL_{ω_1}) by disregarding the dynamic operator. This logic studies *Markov processes with initial distribution*, i.e. structures of the form $\langle \Omega, \mathcal{A}, T, \pi \rangle$ where $\langle \Omega, \mathcal{A}, T \rangle$ is a Markov process and $\pi : \mathcal{A} \to [0, 1]$ is a σ-additive probability measure. We show that the strong expressive power of InPL_{ω_1} would allow us to define *n-step transition probabilities* T^n of Markov kernel T. From this, we conclude that many natural stochastic properties of Markov processes such as stationary, invariance, irreducibility, and recurrence can be stated within InPL_{ω_1}. These results particularly show that DPL as well as DPL_{ω_1} are natural and important extensions of PL.

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