On continuity and openness of maps between locales

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Let $f\colon X \to Y$ be a function between topological spaces (or, more generally, closure spaces) and define the pair of assignments

$$\begin{tikzcd} \mathrm{Closed}(X) \arrow[r, bend left, "f^{\rightarrow}"] & \mathrm{Closed}(Y) \arrow[l, bend left, "f^{\leftarrow}"] \end{tikzcd} \qquad \begin{array}{ccc} A & \mapsto & \mathrm{cl}\,(f[A]) \\ \mathrm{cl}\,(f^{-1}[B]) & \mapsfrom & B \end{array}$$

It is very easy to check that

f is continuous iff $(f^{\rightarrow}, f^{\leftarrow})$ is an adjoint pair.

This idea that the continuous maps between topological spaces (and, more generally, closure spaces) correspond to certain adjoint pairs of maps between the involved closure systems, by assigning with any continuous map the lifted map taking the closures of images as left adjoint and the preimage map as right adjoint, is well-known [2, 3, 7]. This idea was also explored in the pointfree setting in the forerunner article [4]. Our goal in this talk is to revisit those adjunctions and to present new characterizations of *continuous* (that is, *localic*) maps and *open* maps (that is, plain maps with *open images of open sublocales* [5, 6]) between locales, in terms of certain Galois adjunctions between the locales of open sublocales or between the colocales of closed sublocales. With these results we can better understand the differences between the morphisms in the classical and pointfree settings.

Let **Loc** denote the category of locales and localic maps ([8]). Recall that a map $f\colon L \to M$ between locales L and M is a *localic map* (the counterpart of a *continuous map* in the pointfree setting) if

(L1) it preserves arbitrary meets (and hence it has a left adjoint $h\colon M \to L$),

(L2) $f(a) = 1 \;\Rightarrow\; a = 1$, and

(L3) $f(h(a) \to b) = a \to f(b)$ for every $a \in M$, $b \in L$.

In a locale L and any $a \in L$, we consider the *open* and *closed* sublocales

$$\mathfrak{o}(a) = \{x \mid a \to x = x\} = \{a \to x \mid x \in L\} \quad \text{and} \quad \mathfrak{c}(a) = {\uparrow} a = \{x \in L \mid x \geq a\}$$

and denote by $\mathfrak{o}L$ and $\mathfrak{c}L$ respectively the sets of open and closed sublocales of L. We start by generalizing the closure and interior operators from sublocales to general subsets and with a discussion of the problems that may emerge from doing so. Then, given a plain map $f\colon L \to M$, we consider $f^*\colon M \to L$ and $f_!\colon L \to M$, given by $f^*(b) = \bigvee\{a \in L \mid f[\mathfrak{o}(a)] \subseteq \mathfrak{o}(b)\}$ and $f_!(a) = \bigvee\{b \in M \mid \mathfrak{o}(b) \subseteq f[\mathfrak{o}(a)]\}$, and the remaining maps in the following diagram, defined by

$$f_{\mathfrak{o}}^{\rightarrow}(\mathfrak{o}(a)) = \neg(\mathrm{cl}\,(f[\mathfrak{c}(a)])), \qquad f_{\mathfrak{o}}^{\leftarrow}(\mathfrak{o}(b)) = \mathrm{int}\,(f^{-1}[\mathfrak{o}(b)]), \qquad f_{\mathfrak{o}}^{\Rightarrow}(\mathfrak{o}(a)) = \mathrm{int}\,(f[\mathfrak{o}(a)]),$$

$$f_{\mathfrak{c}}^{\rightarrow}(\mathfrak{c}(a)) = \mathrm{cl}\,(f[\mathfrak{c}(a)]), \qquad f_{\mathfrak{c}}^{\leftarrow}(\mathfrak{c}(b)) = \mathrm{cl}\,(f^{-1}[\mathfrak{c}(b)]), \qquad f_{\mathfrak{c}}^{\Rightarrow}(\mathfrak{c}(a)) = \neg(\mathrm{int}\,(f[\mathfrak{o}(a)])).$$

*Joint work with Jorge Picado [1].

$$
\begin{tikzcd}
\mathfrak{o}L \arrow[r, "f_{\mathfrak{o}}^{\Rightarrow}"] \arrow[dd, bend left, "\neg"] & \mathfrak{o}M \arrow[r, "f_{\mathfrak{o}}^{\leftarrow}"] \arrow[dd, bend left, "\neg"] & \mathfrak{o}L \arrow[r, "f_{\mathfrak{o}}^{\rightarrow}"] \arrow[dd, bend left, "\neg"] & \mathfrak{o}M \arrow[dd, bend left, "\neg"] \\
L \arrow[u, "\mathfrak{o}"'] \arrow[d, "\mathfrak{c}"'] \arrow[r, "f_!"] & M \arrow[u, "\mathfrak{o}"'] \arrow[d, "\mathfrak{c}"'] \arrow[r, "f^*"] & L \arrow[u, "\mathfrak{o}"'] \arrow[d, "\mathfrak{c}"'] \arrow[r, "f"] & M \arrow[u, "\mathfrak{o}"'] \arrow[d, "\mathfrak{c}"'] \\
\mathfrak{c}L \arrow[uu, bend left, "\neg"] \arrow[r, "f_{\mathfrak{c}}^{\Rightarrow}"] & \mathfrak{c}M \arrow[uu, bend left, "\neg"] \arrow[r, "f_{\mathfrak{c}}^{\leftarrow}"] & \mathfrak{c}L \arrow[uu, bend left, "\neg"] \arrow[r, "f_{\mathfrak{c}}^{\rightarrow}"] & \mathfrak{c}M \arrow[uu, bend left, "\neg"]
\end{tikzcd}
$$

One has:

1. The pair $(f_{\mathfrak{c}}^{\rightarrow}, f_{\mathfrak{c}}^{\leftarrow})$ is an adjoint pair if and only if f is meet-preserving.
2. If f is order-preserving then the pair $(f_{\mathfrak{o}}^{\leftarrow}, f_{\mathfrak{o}}^{\rightarrow})$ is an adjoint pair if and only if f is a localic map.
3. If f is meet-preserving then:
   (a) The pair $(f_{\mathfrak{o}}^{\Rightarrow}, f_{\mathfrak{o}}^{\leftarrow})$ is an adjunction if and only if f is open.
   (b) The pair $(f_{\mathfrak{c}}^{\leftarrow}, f_{\mathfrak{c}}^{\Rightarrow})$ is an adjunction if and only if f is an open localic map.

An attractive feature of these adjunctions is that they are all concerned with elementary ideas and basic concepts of localic topology: the use of the concrete language of sublocales and its technique simplifies the reasoning. Taking advantage of the generalization of the interior operator and of the characterization of localic maps in [4], we further obtain the following results:

Proposition 1. *A plain map $f\colon L \to M$ is a localic map if and only if*

$$\neg(\operatorname{int}(f^{-1}[\mathfrak{o}(b)])) = f^{-1}[\mathfrak{c}(b)] \quad \textit{for every } b \in M.$$

Proposition 2. *A plain map $f\colon L \to M$ is an open localic map if and only if*

$$\neg(\operatorname{int}(f^{-1}[T])) = f^{-1}[\neg(\operatorname{int} T)] \quad \textit{for every sublocale } T \textit{ of } M.$$

If time permits we will also refer to some open questions.

References

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