A 2-categorical analysis of context comprehension

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The problem of modelling the structural rules of type dependency using categories has motivated the study of several structures, varying in generality, occurrence in nature, and adherence to the syntax of dependent type theory. One aspect, that involving free variables and substitution, is neatly dealt with using (possibly refinements of) Grothendieck fibrations. The other main aspect of type dependency is the possibility of making assumptions as encoded in the two rules below

$$\frac{\Gamma \vdash A \ \mathtt{Type}}{\vdash \Gamma.A \ \mathtt{ctx}} \qquad\qquad \frac{\Gamma \vdash A \ \mathtt{Type}}{\Gamma.A \vdash \mathtt{v}_A : A}$$

where the first one (*context extension*) extends the context Γ with the type A, and the second one (*assumption*) provides a "generic term" of A in context $\Gamma.A$. In the first order setting, they allow us to add assumptions to a context, and to prove what has been assumed, respectively.

We present a purely 2-categorical comparison of the two main categorical accounts of these two rules: Jacobs' comprehension categories [Jac99] and Dybjer's categories with families [Dyb96]. They differ in that the former gives prominence to context extension, and the latter to assumption. The comparison itself consists of a biequivalence of 2-categories, which generalises the classical 1-equivalence between the discrete versions of these structures due to Hofmann [Hof97].

The biequivalence goes via a third 2-category of a less known structure called *weakening and contraction comonad*. These appear already in [Jac99, Definition 9.3.1], where Jacobs uses them to justify the definition of comprehension category [Jac99, Theorem 9.3.4]. We call them *w-comonads* for short. On the other hand, categories with families can be formulated as a pair of discrete fibrations over the same base connected by a (suitable) adjunction. This is known thanks to the observations (and proofs) of, among others, Fiore [Fio08], Awodey [Awo18], and Uemura [Uem23, Section 3]. In order to have a uniform comparison with comprehension categories, we drop the assumption of discreteness on the two fibrations and call the resulting structure a *generalised category with families*.

Morphisms of these structures can vary according to the degree of preservation of the relevant structure. We use the well-established taxonomy of morphisms of adjunctions and of (co)monads [KS74, Str72] to classify morphisms of comprehension categories and of generalised categories with families according to the degree of preservation of context comprehension. In particular, this classification entails that there is a single notion of morphism of which all those that have appeared in the literature are particular cases.

Categories with families are in bijection with discrete comprehension categories because, for every object A of $\mathcal{U}$, the objects of $\dot{\mathcal{U}}$ mapped to A (the terms) are in bijection with

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Figure 1: The underlying diagrams in **Cat** of, from left to right, a comprehension category, a w-comonad, and a generalised category with families.

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\chi"] \arrow[dr, "u"'] & & \mathcal{B}^2 \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{B} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{U} \arrow[loop left, "K"] \arrow[d, "u"] \\ \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \dot{\mathcal{U}} \arrow[rr, bend right, "\Sigma"'] \arrow[dr, "\dot{u}"'] & \top & \mathcal{U} \arrow[ll, bend right, "\Delta"'] \arrow[dl, "u"] \\ & \mathcal{B} & \end{tikzcd}$$

sections of the display map χA. In general, sections can be described as coalgebras, and these specific sections are the coalgebras of the w-comonad K induced by χ. This simple observation suggests that the classical correspondence between categories with families and comprehension categories could be phrased within the framework of the correspondence between adjunctions and comonads. The structure-semantics adjunction [Dub70, Str72] can be used to show that comonads are 2-reflective in a suitable 2-category of adjunctions, where the 1-cells are pairs of functors commuting with the left adjoints. Of course, this reflection is in general far from being an equivalence. Nevertheless, we show that it lifts to a 2-reflection between generalised categories with families and w-comonads which becomes a biequivalence if one takes as morphisms of generalised categories with families functors that commute with left adjoints up to a natural vertical isomorphism. We call these *loose* morphisms. In type theoretic terms, this means preserving typing only up to (vertical) isomorphism. The equivalence in the discrete case is recovered thanks to the fact that vertical isomorphisms in discrete fibrations are identities.

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