Distributive lattice-ordered pregroups

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Abstract

We show that the variety of distributive ℓ-pregroups is generated by a single functional algebra, $\mathbf{F}(\mathbb{Z})$, and that it has a decidable equational theory. We also prove generation and decidability results for each of its n-periodic subvarieties.

1 Introduction

A *lattice-ordered pregroup* (*ℓ-pregroup*) is an algebra $(A, \wedge, \vee, \cdot, {}^{\ell}, {}^{r}, 1)$, where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, multiplication preserves the lattice order $\leq$, and for all x,

$$x^{\ell}x \leq 1 \leq xx^{\ell} \text{ and } xx^{r} \leq 1 \leq x^{r}x.$$

We often refer to x^{ℓ} and x^{r} as the *left* and *right inverse* of x, respectively. The well-studied lattice-ordered groups (ℓ-groups) are exactly the ℓ-pregroups where the two inverses coincide: $x^{\ell} = x^{r}$. Also, ℓ-pregroups constitute lattice-ordered versions of *pregroups*, which are ordered structures introduced by Lambek [11] in the study of applied linguistics, where they are used to describe sentence patterns in many natural languages; they have also been studied extensively by Buzkowski [1] and others in the context of mathematical linguistics in connection to context-free grammars. Pregroups where the order is discrete are exactly groups.

The main reason for our interest in ℓ-pregroups is that they are precisely the *involutive residuated lattices* that satisfy $x + y = xy$; in that respect their study is connected to the algebraic semantics of *substructural logics* [6].

It is easy to show that the underlying lattices of ℓ-groups are distributive. In [5] we show that ℓ-pregroups are semidistributive, but it remains an open problem whether every ℓ-pregroup is distributive. In this submission we focus on the variety DLP of *distributive ℓ-pregroups*.

In analogy to Cayley's theorem for groups, Holland's *embedding theorem* [9] shows that every ℓ-group can be embedded into a symmetric ℓ-group $\mathbf{Aut}(\boldsymbol{\Omega})$—the group of order-preserving permutations on a totally ordered set $\boldsymbol{\Omega}$. Also, Holland's *generation theorem* [10] states that $\mathbf{Aut}(\mathbb{Q})$ generates the variety of ℓ-groups and this is further used to show that the equational theory of ℓ-groups is decidable. In [2] we showed that every distributive ℓ-pregroup embeds into a functional ℓ-pregroup $\mathbf{F}(\boldsymbol{\Omega})$ (a generalization of a symmetric ℓ-group), where $\boldsymbol{\Omega}$ is a chain.

In this submission, which is based on [7], we improve this embedding theorem by showing that every distributive ℓ-pregroup embeds into $\mathbf{F}(\boldsymbol{\Omega})$, where $\boldsymbol{\Omega}$ is an ordinal sum of copies of the integers (we call such chains *integral*). This allows us to obtain an analogue of Holland's generation theorem: the ℓ-pregroup $\mathbf{F}(\mathbb{Z})$ generates the variety DLP. Furthermore, we use this result to prove the decidability of the equational theory of distributive ℓ-pregroups. The methods we use are based on the notion of *diagram*, which is a finitistic object that captures the failure of an equation. The diagrams situation in ℓ-pregroups is much more complex than in ℓ-groups, as one-sided inverses can pile up and computating them in a diagram is quite involved.

Time permitting, we will also discuss our work included in [8]. For every positive integer n, the functions f in $\mathbf{F}(\mathbb{Z})$ that are periodic and have period n end up being exactly the ones that satisfy $f^{\ell^n} = f^{r^n}$; in particular, the ones satifying $f^\ell = f^r$ are the order-preserving permutations on $\mathbb{Z}$. Taking this as inspiration, an element x in an ℓ-pregroup is called *n-periodic* if $x^{\ell^n} = x^{r^n}$; an ℓ-pregroup is called *n-periodic* if all of its elements are, and the corresponding variety is denoted by LP_n. In [3] we showed that $\mathsf{LP}_n \subseteq \mathsf{DLP}$, for all n. Using *n-periodic* diagrams we prove that the join of all of the LP_n's is exactly DLP; this is the analogue of the corresponding theorem for the variety of involutive residuated lattices that we proved in [4] using proof-theoretic methods.

Moreover, we get a representation theorem: every algebra in LP_n can be embedded in the subalgebra $\mathbf{F}_n(\mathbf{\Omega})$ of n-periodic elements of $\mathbf{F}(\mathbf{\Omega})$, for an integral chain $\mathbf{\Omega}$. We prove that DLP is also equal to the join of the varieties $\mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$, thus $\bigvee \mathsf{LP_n} = \bigvee \mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$, but unfortunately $\mathsf{LP_n} \neq \mathsf{V}(\mathbf{F}_n(\mathbb{Z}))$ for every single n. By [10], $\mathsf{LP}_1 = \mathsf{V}(\mathbf{F}_1(\mathbb{Q}))$, but we show that $\mathsf{LP}_n \neq \mathsf{V}(\mathbf{F}_n(\mathbb{Q}))$, for all $n > 1$. In the end we find suitable chains $\mathbf{\Omega}_n$, such that $\mathsf{LP}_n = \mathsf{V}(\mathbf{F}_n(\mathbf{\Omega}_n))$, for every n; actually, we do better than that by identifying a single uniform chain: $\mathsf{LP}_n = \mathsf{V}(\mathbf{F}_n(\mathbb{Q} \overrightarrow{\times} \mathbb{Z}))$, for all n. This result is obtained by a deep analysis of the structure of n-periodic ℓ-pregroups. We prove that every such algebra can be embedded in a *wreath product* of an ℓ-group and $\mathbf{F}_n(\mathbb{Z})$, we analyze the global and local components and see how this is reflected on n-periodic *partition diagrams*.

We also prove that for every n, the equational theories of LP_n and of $\mathbf{F}_n(\mathbb{Z})$ are decidable, where the latter plays a crucial role for the former. The height (difference between input and output values) of a function in $\mathbf{F}_n(\mathbb{Z})$ involved in a failure of an equation needs to be controlled in order to obtain decidability. We show that functions in $\mathbf{F}_n(\mathbb{Z})$ decompose into translations and functions of short height. We use results from linear algebra to control the height of the automorphism part and compose this short piece back to obtain a new short function of $\mathbf{F}_n(\mathbb{Z})$.

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